

# SUPER-POINCARÉ ALGEBRAS, SPACE-TIMES AND SUPERGRAVITIES (II)

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ABSTRACT. The presentation of supergravity theories of our previous paper “Super-Poincaré algebras, space-times and supergravities (I)” is re-formulated in the language of Berezin-Leites-Kostant theory of supermanifolds. It is also shown that the equations of Cremmer, Julia and Scherk’s theory of 11D-supergravity are equivalent to manifestly covariant equations on a supermanifold.

## 1. INTRODUCTION

In a previous article ([32]), we proposed formulations of supergravity theories, based on the notion of an extended space-time  $(M, M_o, \mathcal{D})$ , formed by a superspace  $M$ , a distinguished submanifold  $M_o \subset M$  (representing the physical space-time) and a non-integrable distribution  $\mathcal{D}$ , with properties determined by imposed supersymmetries of vacuum solutions. In such formulations, a supergravity theory is represented by a collection of tensor fields and tensorial equations on  $M$ , whose restrictions at the points of  $M_o$  give the physical fields and their equation of motion.

Many conceptual ingredients of our approach, such as the notion of a superspace  $M$  and of tensorial equations on a superspace, are standard. But, at the best of our knowledge, a presentation of all material in a coordinate-free language and in terms of classical differential geometric objects was still missing. So, we developed such presentation pursuing the following leading intent: *Get an economical description of supergravity theories in terms of objects, which can be studied with standard techniques of Differential Geometry.*

Let us briefly recall the main points of our presentation in [32].

Given a Poincaré algebra  $\mathfrak{p} = \mathfrak{so}(V) + V$  of a flat pseudo-Riemannian space  $V = \mathbb{R}^{p,q}$  and a (super or  $\mathbb{Z}_2$ -graded) extended algebra  $\mathfrak{g} = \mathfrak{so}(V) + V + S$ , we call *space-time of type  $\mathfrak{g}$*  any (super-)manifold  $M$ , together with

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a distinguished submanifold  $M_o \subset M$  and a non integrable distribution  $\mathcal{D}$ , whose Levi form  $\mathcal{L}$  is modeled on the Lie brackets of elements in  $S \subset \mathfrak{g}$ .

A *gravity field on  $M$*  is a pair  $(g, \nabla)$ , formed by a tensor field  $g$  of type  $(0, 2)$ , inducing a pseudo-Riemannian metric on the  $g$ -orthogonal distribution  $\mathcal{D}^\perp$ , and a covariant derivation  $\nabla$  preserving  $\mathcal{D}$ ,  $g$  and  $\mathcal{L}$ .

A *supergravity of type  $\mathfrak{g}$*  is a pair  $\mathcal{G} = ((M, M_o, \mathcal{D}), (g, \nabla))$ , formed by a space-time  $(M, M_o, \mathcal{D})$  of type  $\mathfrak{g}$  and a gravity field  $(g, \nabla)$ . The *physical fields* of  $\mathcal{G}$  are formed by a pair of covariant derivations on  $M_o$ , called *metric* and *spinor connections*, and by three tensor fields, representing the *graviton*, the *gravitino* and the *auxiliary field(s)*, respectively.

There is a distinguished class of supergravities, the so-called (*strict*) *Levi-Civita supergravities of type  $\mathfrak{g}$* , characterized by the vanishing of some special parts of the torsion  $T$  of  $\nabla$ . The importance of such supergravities comes from an existence and uniqueness theorem, which implies that their physical fields are completely determined by the graviton, the gravitino and the auxiliary field(s), as it is required in the standard component approach to supergravities.

We also recall that in [32], we showed that the variations of graviton and gravitino of a Levi-Civita supergravity, determined by Lie derivatives along vector fields of  $M$ , nicely match the supersymmetric transformation rules of simple 4D-supergravity and other supergravities, determined in component formalism. And one can directly check that the same occurs for the variations of graviton and gravitino in Cremmer-Julia-Scherk 11D-supergravity (§4.4). All this can be considered as a supporting evidence for the idea that (localized) supersymmetric invariance of supergravity theories is actually a sort of *Principle of General Covariance*, i.e. a principle of invariance under local changes of coordinates (or, equivalently, local diffeomorphisms) of the superspace  $M$ .

The purpose of this paper is to rewrite the contents of [32] in terms of well defined notions of supergeometry. We essentially follow the theory developed by Berezin, Leites, Kostant et al., but with a modified notion of supermanifold, which we call  $\mathcal{M}$ -supermanifold (§2.2; see also [33]). A  $\mathcal{M}$ -supermanifold  $M^{\mathcal{M}} = (M_o, \mathcal{A}_M^{\mathcal{M}})$  is characterized by an algebra  $\mathcal{A}_M^{\mathcal{M}}$  of superfunctions, which can be considered as the super-analogue of an algebra on a smooth manifold of smooth functions taking values into a  $\mathbb{Z}_2$ -graded algebra (and hence generated by “even” and “odd-valued” functions). In fact, the need for  $\mathcal{M}$ -valued functions naturally arises in any theory involving fermions (anti-commuting quantum fields), not only supersymmetric ones. In supergravity theories bosons and fermions are necessarily intertwined; there is therefore even more reason to consider  $\mathcal{M}$ -valued superfunctions and  $\mathcal{M}$ -supermanifolds.

After this, we apply our approach to the case of Cremmer, Julia and Scherk’s theory of 11D-supergravity ([14]). More precisely, we show how the fields and equations of 11D-supergravity can be expressed in terms of a

supergravity  $\mathcal{G} = ((M^{\mathcal{M}}, M_o^{\mathcal{M}}, \mathcal{D}), (g, \nabla))$  of type  $\mathfrak{g}$  endowed with a suitable 4-form  $\mathcal{F}$  on  $M^{\mathcal{M}}$ . Using results of [12], we get the existence of a one-to-one correspondence between solutions in component formalism of 11D-supergravity equations and quadruples  $(\mathcal{D}, g, \nabla, \mathcal{F})$  on  $M^{\mathcal{M}}$ , which satisfy a set of constraints and equations of purely tensorial type. Due to this, all questions concerning constructions and analysis of solutions of 11D-supergravity can be naturally reduced to problems on suitable geometric structures on  $\mathcal{M}$ -supermanifolds.

The structure of the paper is the following. In §2, we use tensor products with a suitable exterior algebra  $\mathcal{M} = \Lambda^* W$  to define objects that behave as even/odd valued functions and even/odd vector fields on a classical smooth manifold. In §3, we introduce the definition of “ $\mathcal{M}$ -supermanifold”, which is a supermanifold with an algebra of superfunctions, analogous to the even/odd valued functions on a manifold. First properties of  $\mathcal{M}$ -supermanifolds are given; In particular, we show that the Lie derivatives along (even) super vector fields are related with 1-parameter groups of (local) diffeomorphisms in perfect analogy with the corresponding relation between Lie derivatives and flows of smooth manifolds. In §3, we re-formulate definitions and properties of supergravities of type  $\mathfrak{g}$  in the language of  $\mathcal{M}$ -supermanifolds. In §4, we show how Cremmer, Julia and Scherk’s theory of eleven dimensional supergravity can be encoded as a theory on supergravities of type  $\mathfrak{g}$  and that it satisfies the *generalized Principle of General Covariance*, introduced in [32]. For reader’s convenience, we briefly outline the theory of supermanifolds in the Appendix.

We conclude observing that the generalization of the notion of supermanifolds, considered in this paper, stems naturally from a functorial approach to supergeometry as considered for instance in [24, 19, 30, 31] (see also [18], Lec. 2). We believe that this is indeed the most appropriate approach to supergeometry.

*Notation.* Given a sheaf or bundle  $\pi : \mathcal{A} \rightarrow N$  over a manifold  $N$ , we denote by  $\mathcal{A}|_{\mathcal{U}}$  the restriction of  $\mathcal{A}$  over a subset  $\mathcal{U} \subset N$ . In particular  $\mathcal{A}|_x = \pi^{-1}(x)$  for any  $x \in N$ . The set of global (resp. local) smooth sections of  $\mathcal{A}$  is denoted by  $\Gamma(\mathcal{A})$  (resp.  $\Gamma_{\text{loc}}(\mathcal{A})$ ). The sheaf of germs of sections of a bundle  $\mathcal{A}$  is denoted by *Sheaf*  $\mathcal{A}$ . Given a manifold  $N$ , we denote by  $\mathfrak{X}(N) = \Gamma(TN)$  the class of smooth vector fields and by  $\mathfrak{F}_N$  the sheaf of germs of smooth real functions of  $N$ .

If  $X$  is a derivation of a ring  $R$ , its action on elements  $f \in R$  is denoted either by  $X(f)$  or by  $X \cdot f$ . If  $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$  is a  $\mathbb{Z}_2$ -graded vector space, the parity  $i = 0, 1$  of an homogeneous element  $f \in \mathcal{A}_i$ , is denoted by  $|f| \in \mathbb{Z}_2$ .

We consider Clifford algebras as defined e.g. in [10] and the Clifford product between vectors of the standard basis of  $\mathbb{R}^{p,q}$  is  $e_i \cdot e_j = -2\eta_{ij}$  and not “ $+2\eta_{ij}$ ”, as it is often assumed in Physics. Due to this, our notation for signatures of Clifford algebras is opposite to the one of several other papers.

## 2. $\mathbb{J}$ -VALUED FUNCTIONS ON MANIFOLDS AND $\mathbb{J}$ -SUPERMANIFOLDS

### 2.1. $\mathbb{J}$ -valued functions and $\mathbb{J}$ -valued fields on classical manifolds.

In Quantum Field Theory, fields of bosonic or fermionic particles correspond to commuting or anti-commuting operators on some suitable Hilbert space. This fact forces to represent them as differential geometric objects, constructed with the following defined *graded* functions and tensors.

Given a finite dimensional vector space  $W = \mathbb{K}^N$ ,  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , we denote by  $\mathbb{J} \stackrel{\text{def}}{=} \bigoplus_{i=0}^N \Lambda^i W$  its exterior algebra, with spaces of *even* and *odd* elements given by

$$\mathbb{J}_0 \stackrel{\text{def}}{=} \bigoplus_{i=2p} \Lambda^i W, \quad \mathbb{J}_1 \stackrel{\text{def}}{=} \bigoplus_{i=2p+1} \Lambda^i W.$$

Elements  $\eta \in \mathbb{J}_i$ ,  $i = 0, 1$ , are called *homogeneous of parity*  $|\eta| = i$ .

The consideration of an exterior algebra  $\mathbb{J} = \bigoplus_{i=0}^N \Lambda^i W$  allows constructions of appropriate models for quantum anti-commuting fields, *provided that*  $\dim W$  *is large enough to prevent unwanted cancellations in products*. In this paper, this is actually the only condition we have to worry about and, from now on, *we consider*  $W = \mathbb{K}^N$  *as fixed, with*  $N$  *sufficiently large*.

Let  $M_o$  be a smooth manifold and  $E = E_0 + E_1 \rightarrow M_o$  a real (resp. complex)  $\mathbb{Z}_2$ -graded vector bundle of finite rank, with fiber  $V = V_0 + V_1$ . The subbundles  $\pi_i : E_i \rightarrow M_o$ ,  $i = 0, 1$ , are called *subbundles of even/odd elements*, respectively. We remark that bundles, in which either  $E_1$  or  $E_0$  is trivial, are admissible: We call them *(purely) even* or *(purely) odd bundles*, respectively. Clearly, any ungraded vector bundle  $\pi : F \rightarrow M_o$  might be endowed with parity equal to 0 or 1 and hence considered as an even or odd, according to the needs.

Given  $M_o$  and a  $\mathbb{Z}_2$ -graded vector bundle  $\pi : E = E_0 + E_1 \rightarrow M_o$ , we denote by  $M_o^{\mathbb{J}}$  and  $E^{\mathbb{J}}$  the bundles

$$\pi_o^{\mathbb{J}} : M_o^{\mathbb{J}} \stackrel{\text{def}}{=} \mathbb{J} \times M_o \rightarrow M_o, \quad \pi^{\mathbb{J}} : E^{\mathbb{J}} \stackrel{\text{def}}{=} \mathbb{J} \otimes_{\mathbb{K}} E \rightarrow M_o,$$

with  $\mathbb{J} = \Lambda^* W$  (here  $W = \mathbb{R}^N$  if  $E$  is real and  $W = \mathbb{C}^N$  if  $E$  is complex) and by  $\mathbb{J} \otimes_{\mathbb{K}} E$  the bundle with fibers  $\mathbb{J} \otimes_{\mathbb{K}} E|_x$ . Sections of  $\mathfrak{F}_{M_o}^{\mathbb{J}} \stackrel{\text{def}}{=} \text{Sheaf}(M_o^{\mathbb{J}})$  are called  *$\mathbb{J}$ -valued functions*, while  $E^{\mathbb{J}}$  is called the *lambdafication of  $E$* .

The bundles  $M_o^{\mathbb{J}}$  and  $E^{\mathbb{J}}$  have natural structures of  $\mathbb{Z}_2$ -graded bundles

$$M_o^{\mathbb{J}} = M_o^{\mathbb{J}_0} + M_o^{\mathbb{J}_1}, \quad E^{\mathbb{J}} = (E^{\mathbb{J}})_0 + (E^{\mathbb{J}})_1,$$

where  $M_o^{\mathbb{J}_i} = \mathbb{J}_i \times M_o$ ,  $(E^{\mathbb{J}})_0 = \mathbb{J}_0 \otimes E_0 + \mathbb{J}_1 \otimes E_1$  and  $(E^{\mathbb{J}})_1 = \mathbb{J}_1 \otimes E_0 + \mathbb{J}_0 \otimes E_1$ . The (local) sections of  $M_o^{\mathbb{J}_0}$  (resp.  $M_o^{\mathbb{J}_1}$ ) are called *even* (resp. *odd*) *valued functions*, while the (local) sections of  $(E^{\mathbb{J}})_0$  (resp.  $(E^{\mathbb{J}})_1$ ) are called *even* (resp. *odd*) *sections*.

Given a  $\mathcal{M}$ -function  $f = \sum_{\alpha} \eta_{\alpha} \otimes f_{\alpha} \in \Gamma_{\text{loc}}(M_o^{\mathcal{M}})$  and a section  $X = \sum_{\alpha} \eta_{\alpha} \otimes X_{\alpha} \in \Gamma_{\text{loc}}(E^{\mathcal{M}})$ , we call *evaluations at  $x \in M_o$*  the values

$$f|_x \stackrel{\text{def}}{=} \sum_{\alpha} \eta_{\alpha} f_{\alpha}(x) \in \mathcal{M} \quad , \quad X|_x \stackrel{\text{def}}{=} \sum_{\alpha} \eta_{\alpha} \otimes X_{\alpha}(x) \in E|_x = \mathcal{M} \otimes_{\mathbb{K}} E|_x .$$

*Sheaf* $(E^{\mathcal{M}})$  has a natural structure of locally free sheaf of  $\mathfrak{F}_{M_o}^{\mathcal{M}}$ -moduli, with products between sections of  $\mathfrak{F}_{M_o}^{\mathcal{M}}$  and *Sheaf* $(E^{\mathcal{M}})$  defined by

$$(\eta \otimes f) \cdot (\eta' \otimes e) \stackrel{\text{def}}{=} \eta \eta' \otimes (f \cdot e) ,$$

for any  $\eta, \eta' \in \mathcal{M}$ ,  $f \in \Gamma(\mathfrak{F}_{M_o})$ ,  $e \in \Gamma(E)$ .

Given a smooth map  $\varphi : M_o \rightarrow N_o$ , the pull-back  $\varphi^* : \mathfrak{F}_{N_o} \rightarrow \mathfrak{F}_{M_o}$  induces a corresponding morphism of sheaves of  $\mathcal{M}$ -moduli  $\varphi^* : \mathfrak{F}_{N_o}^{\mathcal{M}} \rightarrow \mathfrak{F}_{M_o}^{\mathcal{M}}$ . Similarly, any even bundle morphism  $\Phi : E \rightarrow E'$ , induces an even  $\mathcal{M}$ -linear bundle morphism  $\Phi : E^{\mathcal{M}} \rightarrow E'^{\mathcal{M}}$ .

**Definition 2.1.** Given an open subset  $\mathcal{U}_o \subset M_o$ , we call *frame field of  $E^{\mathcal{M}}$  on  $\mathcal{U}_o$*  any collection  $(e_1, \dots, e_r)$  of homogeneous  $e_i \in \Gamma(E|_{\mathcal{U}_o})$  such that:

- $(e_1|_x, \dots, e_r|_x)$  is a basis of the  $\mathcal{M}$ -module  $E|_x$  for any  $x \in \mathcal{U}_o$ ;
- there is an integer  $r_o \in \mathbb{N} \cup \{0\}$ , such that the  $e_i$ 's, with  $1 \leq i \leq r_o$ , are in  $\Gamma((E|_{\mathcal{U}_o})_0)$ , while the  $e_j$ 's, with  $r_o + 1 \leq j \leq r$ , are in  $\Gamma((E|_{\mathcal{U}_o})_1)$ .

The elements  $(e_1, \dots, e_{r_o})$  (resp.  $(e_{r_o+1}, \dots, e_r)$ ) are called *even* (resp. *odd*) *elements of the frame field*.

The number of even and odd elements of a local frame field does not depend on the choice of the local frame, due to the following simple lemma, whose proof is left to the reader.

**Lemma 2.2.** *For any  $x \in M_o$ ,  $\dim_{\mathcal{M}} E|_x = \dim_{\mathbb{K}} E|_x$  and the number of even elements of a frame field  $(e_i)$  is  $r_o = \dim_{\mathbb{K}} E_0|_x$ .*

Local frames allow the definition of the following subbundles. Given  $0 \leq p_o \leq r_o$ ,  $0 \leq q_o \leq r - r_o$ , a subbundle  $F \subset E^{\mathcal{M}}$  is called *regular of bi-rank  $(p_o, q_o)$*  if any  $x_o \in M_o$  admits a neighborhood  $\mathcal{U}_o$  and a frame field  $(e_i)$  on  $\mathcal{U}_o$ , such that

$$F|_x = \text{Span}_{\mathcal{M}}(e_j|_x, j \in \{1, \dots, p_o\} \cup \{r_o + 1, \dots, r_o + q_o\}) , \quad x \in \mathcal{U}_o .$$

Coming back to tuples  $(\psi^{\alpha})$  of anti-commuting operators, which locally represent fermions (as e.g. the components of a Dirac field), we may conveniently identify them with odd-valued components of sections  $\psi : \mathcal{U} \rightarrow E^{\mathcal{M}}$  of a suitable lambdification  $E^{\mathcal{M}}$ . The parity of  $\psi$  depends on the parity of the elements of the frame field. In case  $E$  is decreed to be purely even (resp. odd), the frame fields has only even (resp. odd) elements and  $\psi$  has clearly odd-valued components  $\psi^{\alpha}$  if and only if it is odd (resp. even). Both alternatives can be handled in equivalent ways, but we found the second one easier-to-use. So, from now on, we adopt the following conventions.

- A lambdification  $E^{\mathcal{M}}$  is called *bosonic* (resp. *fermionic*) if the underlying vector bundle  $E$  is purely even (resp. odd); A subbundle  $F \subset E^{\mathcal{M}}$  of a lambdification  $E^{\mathcal{M}}$  is called *bosonic* (resp. *fermionic*) if it is regular of bi-rank  $(p_o, 0)$  (resp.  $(0, q_o)$ );
- We call *fields in  $E^{\mathcal{M}}$*  the *even* sections  $\psi : \mathcal{U} \rightarrow (E^{\mathcal{M}})_0$ . A field is called *bosonic* (resp. *fermionic*) if and only if it is an even section of a bosonic (resp. fermionic) bundle.

We conclude with the notion of “conjugation”. Following a common habit in Physics ([12], p. 336), we call *standard conjugation of  $\mathcal{M} = \Lambda^*W$* ,  $W = \mathbb{C}^N$ , the unique  $\mathbb{C}$ -anti-linear involution, which coincides on  $W$  with the usual conjugation and is an algebra anti-homomorphism, i.e. such that  $\overline{w_1 \cdot w_2} = \overline{w_2} \cdot \overline{w_1}$  for any  $w_i \in \mathcal{M}$ . We use the symbol “ $\bar{\phantom{x}}$ ” to denote also the associated standard conjugation of (local) sections of  $M_o^{\mathcal{M}} = \mathcal{M} \times M_o$ .

If  $\pi : E \rightarrow M_o$  is a complex vector bundle, endowed with a  $\mathbb{C}$ -antilinear, fiber bundle involution  $\bar{\phantom{x}} : E \rightarrow E$ , we use the same symbol “ $\bar{\phantom{x}}$ ” to denote the induced involution on  $E^{\mathcal{M}}$ , defined by

$$\overline{f \cdot e} = (-1)^{|f||e|} \bar{f} \cdot \bar{e} \text{ for any homogeneous } f \in \Gamma_{\text{loc}}(\mathfrak{F}_{M_o}^{\mathcal{M}}), e \in \Gamma_{\text{loc}}(E).$$

**Example 2.3.** We want to show how Dirac’s Lagrangian for free electrons (see e.g. [9]) can be defined using the objects of this section. Let  $\pi : \mathcal{S} = S \times \mathbb{R}^{3,1} \rightarrow \mathbb{R}^{3,1}$  be the (trivial) spinor bundle of  $\mathbb{R}^{3,1}$ , with fiber given by the space of Dirac spinors  $S = \mathbb{C}^4$ . We consider  $\mathcal{S}$  as a purely odd bundle, so that electrons (which are fermions!) are represented by fields (= even sections)  $\psi : \mathcal{U} \subset \mathbb{R}^{3,1} \rightarrow \mathcal{S}^{\mathcal{M}}$  in the fermionic bundle  $\mathcal{S}^{\mathcal{M}}$ . Dirac’s Lagrangian can be considered as the map  $\mathcal{L} : \Gamma_{\text{loc}}(\mathcal{S}^{\mathcal{M}}) \rightarrow \Gamma_{\text{loc}}((\Lambda^4 T^* \mathbb{R}^{3,1})^{\mathcal{M}})$  given by

$$\mathcal{L}(\psi) = \{i \langle \bar{\psi}, \Gamma^j \cdot D_{e_j} \psi \rangle - m \langle \bar{\psi}, \psi \rangle\} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3,$$

where:

- $D$  is the flat Levi-Civita connection of  $\mathbb{R}^{3,1}$  and  $(e_i)$  is the standard orthonormal basis of  $\mathbb{R}^{3,1}$ , with  $\langle e_i, e_j \rangle = \eta_{ij} = \epsilon_i \delta_{ij}$  where  $\epsilon_0 = -1$  and  $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ ;
- $\Gamma^j : \mathcal{S}^{\mathcal{M}} \rightarrow \mathcal{S}^{\mathcal{M}}$  are the  $\mathcal{M}$ -linear bundle maps, represented in standard frames by the classical Dirac matrices  $\Gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$  and  $\Gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}$  for  $j \neq 0$ , with  $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ ,  $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ;
- $\langle \cdot, \cdot \rangle : \mathcal{S}^{\mathcal{M}} \times \mathcal{S}^{\mathcal{M}} \rightarrow \mathcal{M}$  is the  $\mathcal{M}$ -bilinear map along fibers (symmetric on even sections), defined by  $\langle s_1, s_2 \rangle|_x = s_1^T|_x \cdot \Gamma^0 \cdot s_2|_x$  and  $m$  is the inertial mass of the electron.

## 2.2. $\mathcal{M}$ -supermanifolds.

We recall that there exist two distinct approaches to the notion of “supermanifold”, one developed by Berezin, Bernstein, Leites, Kostant and others

([6, 7, 8, 22, 20, 23]) and another invented by DeWitt, Batchelor and Rogers ([16, 4, 5, 27, 28, 3, 29]). These approaches turn out to be equivalent if some technical modifications and adjustments of basic definitions are considered (see [5, 28, 30]).

We follow Berezin-Leites-Kostant approach, of which the reader can find all main definitions and properties in the appendix. But we have to stress the fact that, in order to reach a satisfactory and rigorous treatment of bosons and fermions, one has to consider supermanifolds with analogues of the even/odd valued functions defined in §2.1. This forces to adopt the following modification of Berezin-Leites-Kostant definition of supermanifold (see also [33, 19]). Notice that:

- such modification essentially coincides with the adjustment that makes Berezin-Leites-Kostant approach equivalent to the Batchelor-DeWitt-Rogers approach and
- it is motivated also by Molotkov's categorical approach ([24, 30]), originally developed to determine a canonical embedding of the group of diffeomorphisms of a supermanifold into a (infinite-dimensional) supergroup.

**Definition 2.4.** Let  $M = (M_o, \mathcal{A}_M)$  be a (real) supermanifold of dimension  $(n|m)$  and  $\mathcal{M} = \Lambda^* W$  the exterior algebra of a fixed vector space  $W = \mathbb{R}^N$ . The corresponding  $\mathcal{M}$ -supermanifold is the pair  $M^{\mathcal{M}} = (M_o, \mathcal{A}_M^{\mathcal{M}})$ , formed by the body  $M_o$  of  $M$  and the sheaf

$$\pi^{\mathcal{M}} : \mathcal{A}_M^{\mathcal{M}} = \mathcal{M} \otimes_{\mathbb{R}} \mathcal{A}_M \longrightarrow M_o ,$$

generated by tensor products  $\eta \otimes \mathfrak{f}$ , with  $\eta \in \mathcal{M}$  and  $\mathfrak{f} \in \Gamma_{\text{loc}}(\mathcal{A}_M)$ . Sections of  $\mathcal{A}_M^{\mathcal{M}}$  are called  $\mathcal{M}$ -superfunctions<sup>(1)</sup>.

The sheaf  $\mathcal{A}_M^{\mathcal{M}}$  is tightly related with the sheaf of superfunctions of a particular Cartesian product of supermanifolds (see §A.1.2). In fact,  $\mathcal{M} = \Lambda^* W$  is isomorphic to the structure sheaf of  $\mathbb{R}^{0|N} = (\{0\}, \mathcal{A}_{\mathbb{R}^{0|N}} = \Lambda^* \mathbb{R}^N)$  and  $\pi^{\mathcal{M}} : \mathcal{A}_M^{\mathcal{M}} \longrightarrow M_o$  is naturally identifiable with the sheaf of superfunctions  $\pi : \mathcal{A}_{\mathbb{R}^{0|N} \times M} \longrightarrow \{0\} \times M_o \simeq M_o$  of the supermanifold  $\mathbb{R}^{0|N} \times M$ . So, we may say that *any  $\mathcal{M}$ -supermanifold  $M^{\mathcal{M}} = (M_o, \mathcal{A}_M^{\mathcal{M}})$  is naturally identifiable with a Cartesian product of supermanifolds of the form  $M^{\mathcal{M}} = \mathbb{R}^{0|N} \times M$* . With the help of such identification, we may introduce the following fundamental objects.

Let  $\mathcal{U} = (\mathcal{U}_o, \mathcal{A}_{M|\mathcal{U}_o})$  be a decomposable neighborhood and  $\xi = (x^i) : \mathcal{U}_o \longrightarrow \mathcal{U}'_o \subset \mathbb{R}^n$  coordinates on  $\mathcal{U}_o$ , associated with a system of supercoordinates

$$\widehat{\xi} : \text{Sheaf}(\Lambda \mathbb{R}^{m*} \times \mathcal{U}'_o) \longrightarrow \mathcal{A}_{M|\mathcal{U}_o} ,$$

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<sup>1</sup>A similar definition can be given for complex  $\mathcal{M}$ -supermanifolds, where  $M = (M_o, \mathcal{A}_M)$  is assumed to be complex and  $\mathbb{K} = \mathbb{R}$  is replaced by  $\mathbb{K} = \mathbb{C}$  at all places.

shortly denoted by  $\widehat{\xi} = (x^i, \vartheta^\alpha)$ . By means of the unique  $\mathcal{M}$ -linear extension

$$\widehat{\xi}^{\mathcal{M}} : \text{Sheaf}(\Lambda \mathbb{R}^{m*} \times \mathcal{U}'_o)^{\mathcal{M}} \longrightarrow \mathcal{A}_{M|\mathcal{U}_o}^{\mathcal{M}}$$

of  $\widehat{\xi} = (x^i, \vartheta^\alpha)$ , the  $\mathcal{M}$ -superfunctions are identified with sums of the form

$$\mathfrak{f} = \sum_{\alpha_j=0,1} \mathfrak{f}_{\alpha_1 \dots \alpha_m}(x^1, \dots, x^n)(\vartheta^1)^{\alpha_1} \wedge \dots \wedge (\vartheta^m)^{\alpha_m},$$

where  $\mathfrak{f}_{\alpha_1 \dots \alpha_m}(x^1, \dots, x^n)$  are  $\mathcal{M}$ -valued functions, called *components of  $\mathfrak{f}$  in coordinates  $(x^i, \vartheta^\alpha)$* . By definitions,  $\mathfrak{f}$  is even if and only if the components  $\mathfrak{f}_{\alpha_1 \dots \alpha_m}$ , whose indices are such that  $\sum_{i=1}^m \alpha_i$  is even, are even-valued, while the other components are odd-valued. A reversed rule characterizes the components of odd superfunctions.

A *morphism* between  $\mathcal{M}$ -supermanifolds  $M^{\mathcal{M}} \simeq \mathbb{R}^{0|N} \times M$ ,  $M'^{\mathcal{M}} \simeq \mathbb{R}^{0|N} \times M'$  is any morphism of supermanifolds  $(f, \widehat{f})$ , in which the sheaf morphism  $\widehat{f} : \mathcal{A}_{M'}^{\mathcal{M}} \longrightarrow \mathcal{A}_M^{\mathcal{M}}$  is  $\mathcal{M}$ -linear, i.e.  $\widehat{f}(\eta \otimes \mathfrak{f}) = \eta \otimes \widehat{f}(\mathfrak{f})$  for any  $\eta \in \mathcal{M}$ . Due to this, given supercoordinates  $(x^i, \vartheta^\alpha)$ ,  $(y^j, \psi^\beta)$  on  $M$  and  $M'$ , respectively, the morphism  $(f, \widehat{f})$  is completely determined by the expressions of the  $\mathcal{M}$ -superfunctions  $y^j(x, \vartheta) = \widehat{f}(y^j)$  and  $\psi^\beta(x, \vartheta) = \widehat{f}(\psi^\beta)$ .

Let us denote by  $\mathfrak{I}_M^{\mathcal{M}} \subset \mathcal{A}_M^{\mathcal{M}}$  the  $\mathcal{A}_M^{\mathcal{M}}$ -invariant subsheaf, generated by  $1 \otimes_{\mathbb{R}} \mathfrak{I}_M \subset \mathcal{A}_M^{\mathcal{M}}$  (see §A.1.1 for definition of  $\mathfrak{I}_M$ ). It can be checked that  $\mathcal{A}_M^{\mathcal{M}}/\mathfrak{I}_M^{\mathcal{M}}$  is naturally identifiable with the sheaf of  $\mathcal{M}$ -valued functions  $\mathfrak{F}_{M_o}^{\mathcal{M}}$ . The projection

$$\epsilon : \mathcal{A}_M^{\mathcal{M}} \longrightarrow \mathcal{A}_M^{\mathcal{M}}/\mathfrak{I}_M^{\mathcal{M}} \simeq \mathfrak{F}_{M_o}^{\mathcal{M}}$$

is called *evaluation map of  $M^{\mathcal{M}}$* . For any  $\mathfrak{f} \in \Gamma_{\text{loc}}(\mathcal{A}_M^{\mathcal{M}})$ , we use the symbols “ $\mathfrak{f}|_{M_o}$ ” to denote  $\epsilon(\mathfrak{f})$  and for any  $x \in M_o$  we use the symbol “ $\mathfrak{f}|_x$ ” to denote the evaluation  $\epsilon(\mathfrak{f})|_x$ . We call *natural embedding of  $M_o^{\mathcal{M}}$  into  $M^{\mathcal{M}}$*  the morphism

$$\iota_{M_o} = (Id_{M_o}, (\cdot)|_{M_o}) : (M_o, \mathfrak{F}_{M_o}^{\mathcal{M}}) \longrightarrow M^{\mathcal{M}} = (M_o, \mathcal{A}_M^{\mathcal{M}})$$

while, for any  $x \in M_o$ , we call *natural embedding of  $x$  into  $M^{\mathcal{M}}$*  the morphism

$$\iota_x = (Id_x, (\cdot)|_x) : (\{x\}, \mathcal{M}) \longrightarrow M^{\mathcal{M}} = (M_o, \mathcal{A}_M^{\mathcal{M}}).$$

In the following, we will use the expression *supervector field of  $M^{\mathcal{M}}$*  to indicate supervector fields of  $M^{\mathcal{M}} \simeq \mathbb{R}^{0|N} \times M$ , which act trivially on any set of odd supercoordinates  $(\eta^1, \dots, \eta^N)$  for  $\mathbb{R}^{0|N}$ . In a system of supercoordinates  $(\eta^\beta, x^i, \vartheta^\alpha)$  of  $\mathbb{R}^{0|N} \times M$ , the supervector fields of  $M^{\mathcal{M}}$  are derivations of the form

$$X = X^j \frac{\partial}{\partial x^j} + X^\alpha \frac{\partial}{\partial \vartheta^\alpha}, \quad \text{with } X^j, X^\alpha \in \Gamma(\mathcal{A}_{M|\mathcal{U}_o}^{\mathcal{M}}),$$

i.e. with vanishing components along the  $\frac{\partial}{\partial \eta^\beta}$ 's.



The sheaf over  $M_o$  of supervector fields of  $M^{\mathbb{J}}$  will be denoted by  $\mathcal{T}M^{\mathbb{J}}$ . Be aware that, by definitions,  $\mathcal{T}M^{\mathbb{J}}$  is a proper subsheaf of the tangent sheaf  $\mathcal{T}(M^{\mathbb{J}}) = \mathcal{T}(\mathbb{R}^{0|N} \times M)$  and that, as  $\mathbb{J}$ -module,

$$\mathcal{T}M^{\mathbb{J}} \simeq \mathcal{A}_M^{\mathbb{J}} \otimes_{\mathcal{A}_M} \mathcal{T}M \simeq \mathbb{J} \otimes_{\mathbb{R}} \mathcal{T}M.$$

For any  $x \in M_o$  and  $X \in \Gamma_{\text{loc}}(\mathcal{T}M^{\mathbb{J}})$ , we call *evaluation of  $X$  at  $x$*  the map

$$X|_x : \mathcal{A}_M^{\mathbb{J}} \longrightarrow \mathbb{J}, \quad X|_x(\mathfrak{f}) \stackrel{\text{def}}{=} (X(\mathfrak{f}))|_x.$$

We call *tangent space of  $M^{\mathbb{J}}$  at  $x$*  the space  $T_x M^{\mathbb{J}}$  generated by the evaluations at  $x$  of supervector fields of  $M^{\mathbb{J}}$ . One can check that  $T_x M^{\mathbb{J}} = (T_x M^{\mathbb{J}})_0 + (T_x M^{\mathbb{J}})_1$ , with

$$(T_x M^{\mathbb{J}})_{\alpha} = \left\{ v : \mathcal{A}_M^{\mathbb{J}} \longrightarrow \mathbb{J}, \quad v(\mathfrak{f} \cdot \mathfrak{g}) = v(\mathfrak{f}) \cdot \mathfrak{g}|_x + (-1)^{\alpha|\mathfrak{f}|} \mathfrak{f}|_x \cdot v(\mathfrak{g}), \right.$$

$$\left. v(\mathfrak{f}) \in \mathbb{J}_{[|\mathfrak{f}|+\alpha]_{\text{mod } 2}} \quad \text{and} \quad v(\eta) = 0 \text{ for any } \eta \in \mathbb{J} \right\}$$

and that  $\mathcal{T}M^{\mathbb{J}}|_{M_o} \stackrel{\text{def}}{=} \bigcup_{x \in M_o} T_x M^{\mathbb{J}}$  is naturally identifiable with the lambda-dification of  $\mathcal{T}M|_{M_o}$ , i.e.  $\mathcal{T}M^{\mathbb{J}}|_{M_o} \simeq (\mathcal{T}M|_{M_o})^{\mathbb{J}}$ .

Let  $\mathcal{U} = (\mathcal{U}_o, \mathcal{A}_{M|\mathcal{U}_o})$  be a decomposable neighborhood with supercoordinates  $\hat{\xi} = (x^i, \vartheta^{\alpha})$ . The sections in  $\Gamma(\text{Sheaf}(\mathcal{T}M^{\mathbb{J}}|_{M_o})|_{\mathcal{U}_o})$  are of the form

$$X = X^j \frac{\partial}{\partial x^j} \Big|_{M_o} + X^{\alpha} \frac{\partial}{\partial \vartheta^{\alpha}} \Big|_{M_o}, \quad X^j, X^{\alpha} \in \Gamma(\mathfrak{F}_{M_o}^{\mathbb{J}}),$$

where for any  $\mathfrak{f} \in \mathcal{A}_M^{\mathbb{J}}$

$$\frac{\partial}{\partial x^i} \Big|_{M_o} \cdot \mathfrak{f} \stackrel{\text{def}}{=} \left( \frac{\partial}{\partial x^i} \cdot \mathfrak{f} \right) \Big|_{M_o} \in \mathfrak{F}_{M_o}^{\mathbb{J}}, \quad \frac{\partial}{\partial \vartheta^{\alpha}} \Big|_{M_o} \cdot \mathfrak{f} \stackrel{\text{def}}{=} \left( \frac{\partial}{\partial \vartheta^{\alpha}} \cdot \mathfrak{f} \right) \Big|_{M_o} \in \mathfrak{F}_{M_o}^{\mathbb{J}}.$$

By definitions,  $X$  is even (resp. odd) if and only if all  $X^j$ 's are even-valued (resp. odd-valued) and all  $X^{\alpha}$ 's are odd-valued (resp. even-valued).

### 2.3. Tensor fields and linear frames on $\mathbb{J}$ -supermanifolds.

For a  $\mathbb{J}$ -supermanifold  $M^{\mathbb{J}} = (M_o, \mathcal{A}_M^{\mathbb{J}})$ , we call *cotangent sheaf of  $M^{\mathbb{J}}$*  the sheaf over  $M_o$  given by

$$\mathcal{T}^* M^{\mathbb{J}} = \text{Hom}_{\mathcal{A}_M^{\mathbb{J}}}(\mathcal{T}M^{\mathbb{J}}, \mathcal{A}_M^{\mathbb{J}}).$$

Local sections of  $\mathcal{T}^* M^{\mathbb{J}}$  are called *1-forms*. A 1-form  $\omega$  is called *homogeneous of parity  $|\omega| \in \mathbb{Z}_2$*  if

$$\omega(\mathcal{T}M_i^{\mathbb{J}}) \subseteq \mathcal{A}_M^{\mathbb{J}}_{[i+|\omega|]_{\text{mod } 2}} \quad \text{and} \quad \omega(\mathfrak{f}X) = (-1)^{|\omega||\mathfrak{f}|} \mathfrak{f}\omega(X)$$

for any homogeneous  $\mathfrak{f} \in \Gamma_{\text{loc}}(\mathcal{A}_M^{\mathbb{J}})$ ,  $X \in \Gamma_{\text{loc}}(\mathcal{T}M^{\mathbb{J}})$ . As for usual supermanifolds, we call *full tensor sheaf of  $M^{\mathbb{J}}$*  the sheaf  $\otimes_{\mathcal{A}_M^{\mathbb{J}}} \mathcal{T}M^{\mathbb{J}}, \mathcal{T}^* M^{\mathbb{J}} >$ , generated by tensor products ( $\mathbb{Z}_2$ -graded over  $\mathcal{A}_M^{\mathbb{J}}$ ) of  $\mathcal{T}M^{\mathbb{J}}$  and  $\mathcal{T}^* M^{\mathbb{J}}$  (see §A.1.3). A local section of  $\otimes_{\mathcal{A}_M^{\mathbb{J}}} \mathcal{T}M^{\mathbb{J}}, \mathcal{T}^* M^{\mathbb{J}} >$  is called *tensor field of type  $(p, q)$*  if it is sum of tensor products of  $p$  vector fields and  $q$  1-forms. The

notions of “homogeneity” and “parity” of tensor fields on  $\mathcal{M}$ -supermanifolds are defined in full analogy with the corresponding notions on supermanifolds. Also the definitions of skew-symmetric tensors of type  $(p, 0)$ ,  $q$ -forms, exterior differential and interior multiplication with vector fields are defined exactly as for usual supermanifolds. See §A.1.4 for a brief review of all such notions and [20] for detailed definitions.

Given a 1-form  $\omega \in \Gamma_{\text{loc}}(\mathcal{T}^*M^{\mathcal{M}})$ , we call *evaluation of  $\omega$  at  $M_o$*  the  $\mathcal{M}$ -linear bundle morphism

$$\omega|_{M_o} : TM^{\mathcal{M}}|_{M_o} \longrightarrow M_o^{\mathcal{M}}, \quad \omega|_{M_o}(X) = \omega(\hat{X})|_{M_o}$$

for any  $X \in \Gamma_{\text{loc}}(TM^{\mathcal{M}}|_{M_o})$  of the form  $X = \hat{X}|_{M_o}$  for some  $\hat{X} \in \Gamma(\mathcal{T}M^{\mathcal{M}})$ . This definition naturally extends to all other tensor fields.

For  $x \in M_o$  and  $\omega \in \Gamma_{\text{loc}}(\mathcal{T}^*M^{\mathcal{M}})$ , the *evaluation of  $\omega$  at  $x$*  is the  $\mathcal{M}$ -linear map

$$\omega|_x : T_x M^{\mathcal{M}} \longrightarrow \mathcal{M}, \quad \omega|_x(X) \stackrel{\text{def}}{=} (\omega(\hat{X}))|_x, \quad \text{with } X = \hat{X}|_x.$$

We call *cotangent space of  $M^{\mathcal{M}}$  at  $x$*  the space  $T_x^*M^{\mathcal{M}}$  generated by the evaluations at  $x$  of 1-forms of  $M^{\mathcal{M}}$ . One can check that  $T_x^*M^{\mathcal{M}} = (T_x^*M^{\mathcal{M}})_0 + (T_x^*M^{\mathcal{M}})_1$  with

$$\begin{aligned} (T_x^*M^{\mathcal{M}})_\alpha = \left\{ \omega : T_x M^{\mathcal{M}} \rightarrow \mathcal{M}, \omega(\eta v) = (-1)^{\alpha|\eta|} \eta \omega(v), \omega(v) \in \mathcal{M}_{[|v|+\alpha] \bmod 2}, \right. \\ \left. \eta \in \mathcal{M}, v \in T_x M^{\mathcal{M}} \right\}. \end{aligned}$$

As for  $TM^{\mathcal{M}}|_{M_o}$ , one can check that  $T^*M^{\mathcal{M}}|_{M_o} = \bigcup_{x \in M_o} T_x^*M^{\mathcal{M}}$  is identifiable with  $(T^*M|_{M_o})^{\mathcal{M}}$ . It follows that

$$(\bigotimes^p TM^{\mathcal{M}}|_{M_o}) \otimes (\bigotimes^q T^*M^{\mathcal{M}}|_{M_o}) \simeq ((\bigotimes^p TM|_{M_o}) \otimes (\bigotimes^q T^*M|_{M_o}))^{\mathcal{M}}.$$

In order to get short statements, close to familiar sentences on smooth manifolds, we consider the following definitions. Given an open subset  $\mathcal{U}_o \subset M_o$ , we call *open subset of  $M^{\mathcal{M}}$*  the  $\mathcal{M}$ -supermanifold  $\mathcal{U}^{\mathcal{M}} = (\mathcal{U}_o, \mathcal{A}_{M|_{\mathcal{U}_o}}^{\mathcal{M}})$  and, if  $x_o \in \mathcal{U}_o$ , we say that  $\mathcal{U}^{\mathcal{M}}$  is a *neighborhood of  $x_o$  in  $M^{\mathcal{M}}$* . Moreover:

**Definition 2.5.** Let  $\mathcal{U}^{\mathcal{M}}$  be an open subset of  $M^{\mathcal{M}}$ . An ordered set  $(e_1, \dots, e_{n+m})$  of supervector fields of  $\mathcal{U}^{\mathcal{M}}$  is called *(local) frame field* if:

- i) it is a collection of  $\mathcal{A}_{M|_{\mathcal{U}_o}}^{\mathcal{M}}$ -linearly independent generators for the  $\mathcal{A}_{M|_{\mathcal{U}_o}}^{\mathcal{M}}$ -module  $\Gamma(\mathcal{T}\mathcal{U}^{\mathcal{M}})$ ;
- ii) all  $e_i$ 's, with  $1 \leq i \leq n$ , are even, while the  $e_j$ 's, with  $n+1 \leq j \leq n+m$ , are odd.

For any open  $\mathcal{U}_o \subset M_o$ , we denote by  $\mathcal{F}r^{\mathcal{M}}(\mathcal{U})$  the collection of linear frames on the corresponding open subset  $\mathcal{U}^{\mathcal{M}} \subset M^{\mathcal{M}}$ . The sheaf  $\pi : \mathcal{F}r^{\mathcal{M}}(M) \longrightarrow M_o$ , determined by the pre-sheaf  $\{\mathcal{U}_o \longrightarrow \mathcal{F}r^{\mathcal{M}}(\mathcal{U})\}$ , is called *sheaf of frame fields of  $M^{\mathcal{M}}$*  (see also [2]). Notice that, for any local

frame field  $(e_1, \dots, e_{n+m})$ , the evaluations  $(e_1|_{M_o}, \dots, e_{n+m}|_{M_o})$  constitute a local frame field for the graded vector bundle  $TM^{\mathcal{M}}|_{M_o} \simeq (TM|_{M_o})^{\mathcal{M}}$ .

#### 2.4. Flows, Lie derivatives and linear connections.

Given a  $\mathcal{M}$ -supermanifold  $M^{\mathcal{M}} = (M_o, \mathcal{A}_M^{\mathcal{M}})$ , we call (*smooth*) *1-parameter group of automorphisms* any morphism of supermanifolds (here, we think of  $M^{\mathcal{M}}$  as a Cartesian product of supermanifolds; see §2.2)

$$(\Phi, \widehat{\Phi}) : ]-a, a[ \times M^{\mathcal{M}} \longrightarrow M^{\mathcal{M}}, \quad a \in \mathbb{R} \cup \{\infty\}$$

such that:

- i) for any  $t \in ]-a, a[$ , the morphism  $(\Phi_t, \widehat{\Phi}_t) : M^{\mathcal{M}} \longrightarrow M^{\mathcal{M}}$  defined by  $\Phi_t \stackrel{\text{def}}{=} \Phi(t, \cdot)$  and  $\widehat{\Phi}_t : \mathcal{A}_M^{\mathcal{M}} \longrightarrow \Phi_{t*}(\mathcal{A}_M^{\mathcal{M}})$ ,  $\widehat{\Phi}_t(\mathfrak{f}) \stackrel{\text{def}}{=} \epsilon_t(\widehat{\Phi}(\mathfrak{f}))$ ,  
is an isomorphism of  $\mathcal{M}$ -supermanifolds;
- ii)  $(\Phi_0, \widehat{\Phi}_0) = \text{Id}_M$  and  $(\Phi_t, \widehat{\Phi}_t) \circ (\Phi_s, \widehat{\Phi}_s) = (\Phi_{t+s}, \widehat{\Phi}_{t+s})$  for any  $t, s$  such that  $t + s \in ]-a, a[$ .

The *supervector field* of  $(\Phi, \widehat{\Phi})$  is the derivation  $V \in \Gamma(\mathcal{T}M^{\mathcal{M}})$  defined by

$$V \cdot \mathfrak{f} = \lim_{h \rightarrow 0} \frac{1}{h} \left( \widehat{\Phi}_h(\mathfrak{f}) - \mathfrak{f} \right) = \left. \frac{d\widehat{\Phi}_t(\mathfrak{f})}{dt} \right|_{t=0}$$

and we say that  $(\Phi, \widehat{\Phi})$  is a *flow* of  $V$ . In analogy with smooth manifolds, it is possible to define *local* 1-parameter groups of automorphisms and corresponding supervector fields. We leave to the reader the task of guessing the appropriate definitions.

One can directly check that the supervector field of a local 1-parameter group of automorphisms is always *even*. The converse is also true. In fact:

**Lemma 2.6** ([25], Thm. 4). *For any even  $V \in \Gamma_{\text{loc}}(\mathcal{T}M^{\mathcal{M}})$ , there exists a local 1-parameter group of automorphisms  $(\Phi, \widehat{\Phi})$ , which is a flow of  $V$ . Given two flows of  $V$ , defined on open subsets  $I \times \mathcal{U}^{\mathcal{M}}$ ,  $I' \times \mathcal{U}^{\mathcal{M}}$ , the isomorphisms  $\widehat{\Phi}_t, \widehat{\Phi}'_t$  coincide for any  $t \in I \cap I'$ .*

For an even supervector field  $V$ , we denote by  $\Phi_t^V$  the corresponding flow. One can check that all properties of local flows on smooth manifolds have analogues for local flows on  $\mathcal{M}$ -supermanifolds. In particular, for any even supervector field  $V$ , the map  $\alpha \mapsto \Phi_{-t*}^V(\alpha)$  on tensor fields  $\alpha$  of  $M^{\mathcal{M}}$  is such that

$$\left. \frac{d\Phi_{-t*}^V(\alpha)}{dt} \right|_{t=0} = \mathcal{L}_V \alpha,$$

where “ $\mathcal{L}_V(\cdot)$ ” denotes the unique derivation of tensors, which is compatible with contractions and such that  $\mathcal{L}_V(\mathfrak{f}) = V \cdot \mathfrak{f}$ ,  $\mathcal{L}_V X = [V, X]$  for any superfunction  $\mathfrak{f}$  and supervector field  $X$ . We call  $\mathcal{L}_V \alpha$  the *Lie derivative of  $\alpha$  along  $V$* . One can check that, writing tensor fields w.r.t. frames  $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \vartheta^\alpha})$  and coframes  $(dx^i, d\vartheta^\alpha)$ , the expression for  $\mathcal{L}_V \alpha$  is identical to the usual formula for Lie derivatives on smooth manifolds.

**Definition 2.7.** A *linear connection* on  $M^{\mathcal{M}}$  is a linear, even map of sheaves of  $\mathcal{M}$ -moduli

$$\nabla : \mathcal{T}M^{\mathcal{M}} \otimes_{\mathbb{R}} \mathcal{T}M^{\mathcal{M}} \longrightarrow \mathcal{T}M^{\mathcal{M}}$$

such that

- 1)  $\nabla_{\mathfrak{f}X}Y = \mathfrak{f}\nabla_XY$ ,
- 2)  $\nabla_X\mathfrak{f}Y = (X\mathfrak{f})Y + (-1)^{|X||\mathfrak{f}|}\mathfrak{f}\nabla_XY$

for any homogeneous  $\mathfrak{f} \in \mathcal{A}_M^{\mathcal{M}}$ ,  $X, Y \in \mathcal{T}M^{\mathcal{M}}$ . The *torsion*  $T$  and the *curvature*  $R$  of a linear connection  $\nabla$  are the tensor fields of type (1, 2) and (1, 3), respectively, such that, for any homogeneous supervector fields  $X, Y, Z \in \Gamma(\mathcal{T}M^{\mathcal{M}})$ ,

$$T_{XY} \stackrel{\text{def}}{=} \nabla_XY - (-1)^{|X||Y|}\nabla_YX - [X, Y] ,$$

$$R_{XY}Z \stackrel{\text{def}}{=} \nabla_X\nabla_YZ - (-1)^{|X||Y|}\nabla_Y\nabla_XZ - \nabla_{[X, Y]}Z .$$

Given a linear connection  $\nabla$ , the *induced connection on the vector bundle*  $\pi : \mathcal{T}M^{\mathcal{M}}|_{M_o} \longrightarrow M_o$  is the map

$$\nabla : \Gamma((\mathcal{T}M_o)^{\mathcal{M}}) \times \Gamma(\mathcal{T}M^{\mathcal{M}}|_{M_o}) \longrightarrow \Gamma(\mathcal{T}M^{\mathcal{M}}|_{M_o}) , \quad \nabla_XY \stackrel{\text{def}}{=} (\nabla_{\hat{X}}\hat{Y})|_{M_o} ,$$

where  $\hat{X}, \hat{Y} \in \Gamma_{\text{loc}}(\mathcal{T}M^{\mathcal{M}})$  are such that  $X = \hat{X}|_{M_o}$ ,  $Y = \hat{Y}|_{M_o}$ .

### 3. SUPER-SPACETIMES AND SUPERGRAVITIES ON $\mathbf{JL}$ -SUPERMANIFOLDS

#### 3.1. Admissible super Poincarè algebras.

Given a flat pseudo-Riemannian space  $V = \mathbb{R}^{p,q}$ , with the scalar product  $\langle \cdot, \cdot \rangle$  and Poincarè algebra  $\mathfrak{p}(V) = \text{Lie}(\text{Iso}(\mathbb{R}^{p,q})) = \mathfrak{so}(V) + V$ , the *super-extensions of  $\mathfrak{p}(V)$*  are the Lie superalgebras  $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$  with the following properties:

- a)  $\mathfrak{g}_0 = \mathfrak{p}(V) = \mathfrak{so}(V) + V$ ;
- b)  $\mathfrak{g}_1 = S$  is an irreducible spinor module (i.e. an irreducible real representation of the Clifford algebra  $\mathcal{Cl}(V)$  of  $V$ ) and the adjoint action  $\text{ad}_{\mathfrak{so}(V)}|_S : S \longrightarrow S$  coincides with the standard action of  $\mathfrak{so}(V)$  on  $S$  (i.e.  $[A, s] = A \cdot s$  for any  $A \in \mathfrak{so}(V)$ ,  $s \in S$ );
- c)  $[V, S] = 0$  and  $[S, S] \subseteq V$ .

Any super-extension of  $\mathfrak{p}(V)$  is called *super Poincarè algebra*.

We recall that, given an irreducible spinor module  $S$ , any super-extension  $\mathfrak{g} = (\mathfrak{so}(V) + V) + S$  is uniquely determined by the  $\mathfrak{so}(V)$ -invariant tensor  $L \in \vee^2 S^* \otimes V$  defining the Lie bracket  $[\cdot, \cdot]_{S \times S} : S \times S \longrightarrow V$  and any  $\mathfrak{so}(V)$ -invariant tensor of this kind gives a super-extension of  $\mathfrak{p}(V)$ .

A tensor  $L \in \vee^2 S^* \otimes V$  is called *admissible* if the associated tensor  $L^* \in \vee^2 S^* \otimes V^*$ , defined by  $L^*(s, s', v) \stackrel{\text{def}}{=} \langle L(s, s'), v \rangle$ , is of the form

$$L^*(s, s', v) = \beta(v \cdot s, s')$$

(here “ $\cdot$ ” stands for Clifford product) for some non-degenerate  $\mathfrak{so}(V)$ -invariant bilinear form  $\beta$  on  $S$  such that:

- 1) it is either symmetric or skew-symmetric;
- 2) the Clifford multiplications  $v \cdot (\cdot) : S \longrightarrow S$ ,  $v \in V$ , are either all  $\beta$ -symmetric or all  $\beta$ -skew-symmetric;
- 3) if  $S$  decomposes into irreducible  $\mathfrak{so}(V)$ -moduli  $S = S^+ + S^-$ , then  $S^\pm$  are either mutually  $\beta$ -orthogonal or both  $\beta$ -isotropic.

It is known that *any admissible tensor is  $\mathfrak{so}(V)$ -invariant, it corresponds to a super Poincaré algebra and there is a basis for  $(\vee^2 S^* \otimes V)^{\mathfrak{so}(V)}$  made of admissible elements ([1]).*

A super Poincaré algebra, determined by an admissible tensor, is called *admissible*. In this case,  $V + S$  is naturally endowed with the non-degenerate bilinear form  $(\cdot, \cdot)$ , called *extended inner product*, defined by

$$(\cdot, \cdot)|_{V \times V} = 0, \quad (\cdot, \cdot)|_{V \times V} = \langle \cdot, \cdot \rangle, \quad (\cdot, \cdot)|_{S \times S} = \beta.$$

From now on, *any super Poincaré algebra is assumed to be admissible and  $(\cdot, \cdot)$  always indicates the extended inner product on  $V + S$ .*

### 3.2. Distributions, Levi forms and super-spacetimes of type $\mathfrak{g}$ .

**Definition 3.1.** A *distribution*  $\mathcal{D}$  on a  $\mathcal{M}$ -supermanifold  $M^{\mathcal{M}}$  is a  $\mathbb{Z}_2$ -graded subsheaf of  $\mathcal{A}_M^{\mathcal{M}}$ -modules of  $\mathcal{T}M^{\mathcal{M}}$ , which is locally a direct factor in  $\mathcal{T}M^{\mathcal{M}}$ . It is called *odd* (resp. *even*) of *rank*  $q$  if for any  $x_o \in M_o$ , there exists neighborhood  $\mathcal{U}^{\mathcal{M}}$  of  $x_o$  in  $M^{\mathcal{M}}$  and a local frame field  $(e_i)$  on  $\mathcal{U}^{\mathcal{M}}$ , such that  $q$  of its odd (resp. even) elements are generators for  $\mathcal{D}|_{\mathcal{U}_o}$ .

The *Levi form* of a distribution  $\mathcal{D}$  is the sheaf morphism

$$\mathcal{L} : \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{T}M^{\mathcal{M}}/\mathcal{D}, \quad (3.1)$$

determined by the map between local vector fields

$$\mathcal{L} : \Gamma_{\text{loc}}(\mathcal{D}) \times \Gamma_{\text{loc}}(\mathcal{D}) \longrightarrow \Gamma_{\text{loc}}(\mathcal{T}M^{\mathcal{M}}/\mathcal{D}), \quad \mathcal{L}(X, Y) \stackrel{\text{def}}{=} [X, Y]/\Gamma_{\text{loc}}(\mathcal{D}).$$

Notice that, if a complementary distribution  $\mathcal{D}^\perp$  of  $\mathcal{D}$  is fixed (possibly only locally defined), the quotient map  $p : \mathcal{T}M^{\mathcal{M}} \longrightarrow \mathcal{T}M^{\mathcal{M}}/\mathcal{D}$  gives a sheaf isomorphism  $p|_{\mathcal{D}^\perp} : \mathcal{D}^\perp \xrightarrow{\sim} \mathcal{T}M^{\mathcal{M}}/\mathcal{D}$  and the Levi form (3.1) is completely determined by the associated tensor field  $\mathcal{L}^{(\mathcal{D}^\perp)}$  of type  $(1, 2)$  defined by

$$\mathcal{L}^{(\mathcal{D}^\perp)}(X, Y) \stackrel{\text{def}}{=} (p|_{\mathcal{D}^\perp})^{-1} \circ (\mathcal{L}(\pi^{\mathcal{D}}(X), \pi^{\mathcal{D}}(Y))) \quad X, Y \in \Gamma_{\text{loc}}(\mathcal{T}M^{\mathcal{M}}),$$

where  $\pi^{\mathcal{D}} : \mathcal{T}M^{\mathcal{M}} \longrightarrow \mathcal{D}$ ,  $\pi^{\mathcal{D}^\perp} : \mathcal{T}M^{\mathcal{M}} \longrightarrow \mathcal{D}^\perp$  are the standard projections, determined by the decomposition  $\mathcal{T}M^{\mathcal{M}} = \mathcal{D} \oplus \mathcal{D}^\perp$ . We call  $\mathcal{L}^{(\mathcal{D}^\perp)}$  the *Levi tensor of  $\mathcal{D}$  determined by  $\mathcal{D}^\perp$* . Whenever the context makes clear which complementary distribution is considered, the symbol  $\mathcal{L}$  will be used to indifferently denote the Levi form and the Levi tensor.

Consider now a super Poincaré algebra  $\mathfrak{g} = (\mathfrak{so}(V) + V) + S$  and a corresponding connected homogeneous superspace  $G/H = (G_o/H, \mathcal{A}_{G/H})$ , with  $\text{Lie}(G_o) = \mathfrak{so}(V) + V$  and  $H \subset G_o$  connected subgroup with  $\mathfrak{h} = \text{Lie}(H) =$

$\mathfrak{so}(V)$ . We call *flat super-spacetime of type  $\mathfrak{g}$*  any  $\mathcal{M}$ -supermanifold  $M^{\mathcal{M}}$  associated with  $M = G/H$ .

Since the subspaces  $V, S \subset \mathfrak{g}$  are  $\text{Ad}_H$ -invariant, the homogeneous superspace  $M = G/H$  and the  $\mathcal{M}$ -supermanifold  $M^{\mathcal{M}}$  admit the complementary  $G$ -invariant distributions  $\mathcal{D}^{\mathfrak{g}}, \mathcal{D}^{\mathfrak{g}\perp}$  defined as follows.

Let  $\mathcal{S}, \mathcal{V} \subset \mathcal{T}G$  be the distributions of  $G$ , generated by the left-invariant vector fields in  $S$  and  $V$ , respectively, and denote by

$$(\pi_o, \hat{\pi}) : G = (G_o, \mathcal{A}_G) \longrightarrow G/H = (G_o/H, \mathcal{A}_{G/H})$$

the natural superspace morphism from  $G$  onto  $G/H$ . Since  $S, V \subset \mathfrak{g}$  are  $\text{Ad}_H$ -invariant, the distributions  $\mathcal{S}, \mathcal{V}$  are invariant under the right action of  $H$  and hence locally generated by vector fields  $s_\alpha \in \mathcal{S}, v_i \in \mathcal{V}$  *invariant under the right-action of  $H$* . For any such vector field, there exists a unique vector field  $\tilde{s}_\alpha$  or  $\tilde{v}_i$  in  $\mathcal{T}G/H$  such that

$$s_\alpha \cdot \hat{\pi}(\mathfrak{f}) = \hat{\pi}(\tilde{s}_\alpha \cdot \mathfrak{f}) \quad \text{or} \quad v_i \cdot \hat{\pi}(\mathfrak{f}) = \hat{\pi}(\tilde{v}_i \cdot \mathfrak{f}) \quad \text{for any } \mathfrak{f} \in \Gamma_{\text{loc}}(\mathcal{A}_{G/H}) .$$

The vector fields  $\tilde{s}_\alpha$  and  $\tilde{v}_i$  generate two complementary  $G$ -invariant distributions in  $\mathcal{T}G/H$  (and in  $\mathcal{T}G/H^{\mathcal{M}}$ ), which we call  $\mathcal{D}^{\mathfrak{g}}$  and  $\mathcal{D}^{\mathfrak{g}\perp}$ , respectively.

Let  $W = (V, \mathcal{A}_W)$  denote the connected super-subgroup of  $G$ , with associated sHC-pair  $(V, V + S)$  (see §A.2). Given two bases  $(e_i)_{i=1, \dots, n}, (e_\alpha)_{\alpha=1, \dots, m}$ , for  $V$  and  $S$  respectively, denote by  $(E_i, E_\alpha)$  the local frame field on  $M^{\mathcal{M}} = (G_o/H, \mathcal{A}_{G/H}^{\mathcal{M}})$ , formed by the  $W$ -invariant even and odd vector fields with

$$E_i|_o = e_i, \quad E_\alpha|_o = e_\alpha, \quad \text{where } o = eH \in G_o/H .$$

One can directly check that:

- a) the  $E_i$ 's are even generators for  $\mathcal{D}^{\mathfrak{g}\perp}$ ;
- b) the  $E_\alpha$ 's are odd generators for  $\mathcal{D}^{\mathfrak{g}}$ ;
- c) the Levi tensor  $\mathcal{L}^{\mathfrak{g}}$  of  $\mathcal{D}^{\mathfrak{g}}$  (determined by  $\mathcal{D}^{\mathfrak{g}\perp}$ ) is such that

$$\mathcal{L}(E_i, E_j) = 0, \quad \mathcal{L}(E_i, E_\alpha) = 0, \quad \mathcal{L}(E_\alpha, E_\beta) = \mathcal{L}_{\alpha\beta}^i E_i, \quad (3.2)$$

where  $\mathcal{L}_{\alpha\beta}^i$  are the structure constants defined by  $[e_\alpha, e_\beta] = \mathcal{L}_{\alpha\beta}^i e_i$ .

Now, consider a super Poincaré algebra  $\mathfrak{g} = (\mathfrak{so}(V) + V) + S$  and let  $n = \dim V, m = \dim S$ .

**Definition 3.2.** A *super-spacetime of type  $\mathfrak{g}$*  is a triple  $(M^{\mathcal{M}}, M_o^{\mathcal{M}}, \mathcal{D})$ , where:

- $M^{\mathcal{M}} = (M_o, \mathcal{A}_M^{\mathcal{M}})$  is  $\mathcal{M}$ -supermanifold of dimension  $(n|m)$ ;
- $M_o^{\mathcal{M}} = \mathcal{M} \times M_o$  where  $M_o$  is the body of  $M^{\mathcal{M}}$ ;
- $\mathcal{D} \subset \mathcal{T}M^{\mathcal{M}}$  is an odd distribution of rank  $m$  satisfying the following “uniformity assumption”: for any  $x_o \in M_o$ , there is a neighborhood  $\mathcal{U}^{\mathcal{M}}$  of  $x_o$  in  $M^{\mathcal{M}}$ , a local frame field  $(E_i, E_\alpha)$  on  $\mathcal{U}^{\mathcal{M}}$  and an associated basis  $(e_i, e_\alpha)$  for  $V + S$  such that
  - i) the odd fields  $E_\alpha$  generate  $\mathcal{D}$  and the even fields  $E_i$  generate a complementary distribution  $\mathcal{D}^\perp$ ;

- ii) the Levi tensor  $\mathcal{L}$  of  $\mathcal{D}$ , determined by  $\mathcal{D}^\perp$ , is such that  $\mathcal{L}(E_\alpha, E_\beta) = \mathcal{L}_{\alpha\beta}^i E_i$ , where  $\mathcal{L}_{\alpha\beta}^i$  are the constants in (3.2);
- iii) if  $S = S^+ + S^-$  is sum of irreducible  $\mathfrak{so}(V)$ -moduli, the basis  $(e_\alpha)$  is formed by two bases  $(e_\beta), (e_{\dot{\beta}})$  for  $S^+, S^-$ , respectively, and the corresponding fields  $E_\beta, E_{\dot{\beta}}$  generate complementary subdistributions  $\mathcal{D}^\pm \subset \mathcal{D}$ .

### 3.3. Supergravities of type $\mathfrak{g}$ .

**Definition 3.3.** A *gravity field* on a super-spacetime  $(M^{\mathcal{M}}, M_o^{\mathcal{M}}, \mathcal{D})$  is a pair  $(g, \nabla)$ , where  $g$  is an even tensor field of type  $(0, 2)$  and  $\nabla$  is a linear connection on  $M^{\mathcal{M}}$  such that

- i) the tensor  $g$  is such that, for any  $x_o \in M_o$ , there is neighborhood  $\mathcal{U}^{\mathcal{M}}$  of  $x_o$  in  $M^{\mathcal{M}}$ , a local frame field  $(E_A) = (E_i, E_\alpha)$  on  $\mathcal{U}$  and an associated basis  $(e_A) = (e_i, e_\alpha)$  for  $V + S$  such that:
  - a) the odd fields  $E_\alpha$  generate  $\mathcal{D}$ , while the even fields  $E_i$  generate a complementary distribution  $\mathcal{D}^\perp$ ; if  $S = S^+ + S^-$ , the  $e_\alpha$ 's are given by bases  $(e_\beta), (e_{\dot{\beta}})$  for  $S^+, S^-$  respectively, and the corresponding fields  $E_\beta, E_{\dot{\beta}}$  generate complementary subdistributions  $\mathcal{D}^\pm \subset \mathcal{D}$ ;
  - b) the fields  $E_A$  are such that  $g(E_A, E_B) \equiv (e_A, e_B)$ , where  $(\cdot, \cdot)$  denotes the extended inner product of  $V + S$ ;
  - c) the components in the frame field  $(E_A)$  of the Levi tensor  $\mathcal{L}$  of  $\mathcal{D}$ , determined by  $\mathcal{D}^\perp$ , are constant and equal to those in (3.2).
- ii) the distribution  $\mathcal{D}$  is  $\nabla$ -stable and, if  $S = S^+ + S^-$ , both distributions  $\mathcal{D}^\pm$  are  $\nabla$ -stable;
- iii)  $\nabla g = 0$  and  $\nabla \mathcal{L} = 0$ .

A *supergravity of type  $\mathfrak{g}$*  is a pair  $\mathcal{G} = ((M^{\mathcal{M}}, M_o^{\mathcal{M}}, \mathcal{D}), (g, \nabla))$  formed by a super-spacetime  $(M^{\mathcal{M}}, M_o^{\mathcal{M}}, \mathcal{D})$  of type  $\mathfrak{g}$  and a gravity field  $(g, \nabla)$  on it.

For a given supergravity  $\mathcal{G} = ((M^{\mathcal{M}}, M_o^{\mathcal{M}}, \mathcal{D}), (g, \nabla))$ , we call *spinor bundle* the fermionic subbundle of  $TM^{\mathcal{M}}|_{M_o}$ , given by  $\mathcal{S} := \mathcal{D}|_{M_o}$ , and we call *physical fields* the following objects:

- the field (=even section)  $\vartheta$  in the fermionic bundle  $T^*M_o \otimes_{M_o} \mathcal{S}$  over  $M_o$ , called *gravitino*, defined by

$$\vartheta(X) \stackrel{\text{def}}{=} \pi^{\mathcal{D}}(\hat{X})|_{M_o} \quad \text{for any } X \in \mathfrak{X}_{\text{loc}}(M_o),$$

where  $\pi^{\mathcal{D}} : TM^{\mathcal{M}} = \mathcal{D} \oplus \mathcal{D}^\perp \longrightarrow \mathcal{D}$  denotes the natural projection onto  $\mathcal{D}$  and  $\hat{X}$  any field in  $\Gamma_{\text{loc}}(TM^{\mathcal{M}})$  with  $\hat{X}|_{M_o} = X$ ;

- the field  $\hat{g}$  in the bosonic bundle  $(\vee^2 T^*M_o)^{\mathcal{M}}$ , called *graviton*, defined by

$$\hat{g}(X, Y) = g(\pi^{\mathcal{D}^\perp}(\hat{X}), \pi^{\mathcal{D}^\perp}(\hat{Y}))|_{M_o} \quad \text{for any } X, Y \in \mathfrak{X}_{\text{loc}}(M_o),$$

- where  $\pi^{\mathcal{D}^\perp} : \mathcal{T}M^{\mathcal{M}} = \mathcal{D} \oplus \mathcal{D}^\perp \longrightarrow \mathcal{D}^\perp$  is the natural projection onto  $\mathcal{D}^\perp$  and  $\widehat{X}, \widehat{Y}$  are fields in  $\Gamma_{\text{loc}}(\mathcal{T}M^{\mathcal{M}})$  with  $\widehat{X}|_{M_o} = X, \widehat{Y}|_{M_o} = Y$ ;
- the field  $\mathbb{A}$  in the bosonic bundle  $T^*M_o \otimes_{M_o} \mathcal{S}^* \otimes_{M_o} \mathcal{S}$ , called *A-field*, defined by

$$\mathbb{A}_{Xs} \stackrel{\text{def}}{=} -\pi^{\mathcal{D}}(T_{\widehat{X}\widehat{s}})|_{M_o} \quad \text{for any } X \in \mathfrak{X}_{\text{loc}}(M_o), s \in \Gamma_{\text{loc}}(\mathcal{S}),$$

- where  $T$  is the torsion of  $\nabla$  and  $\widehat{X}, \widehat{s}$  are fields in  $\Gamma_{\text{loc}}(\mathcal{T}M^{\mathcal{M}})$  such that  $\widehat{X}|_{M_o} = X, \widehat{s}|_{M_o} = s$ ;
- the connection  $D : \mathfrak{X}(M_o) \times \mathfrak{X}(M_o) \longrightarrow \Gamma((TM_o)^{\mathcal{M}})$ , called *metric connection*, defined by

$$D_X Y \stackrel{\text{def}}{=} (\pi^{\mathcal{D}^\perp}|_{\Gamma((TM_o)^{\mathcal{M}})})^{-1} \left( \nabla_{\widehat{X}} \left( \pi^{\mathcal{D}^\perp}(\widehat{Y}) \right) \right)|_{M_o}$$

- where  $\widehat{X}, \widehat{Y}$  are fields in  $\Gamma_{\text{loc}}(\mathcal{T}M^{\mathcal{M}})$  such that  $\widehat{X}|_{M_o} = X, \widehat{Y}|_{M_o} = Y$ ;
- the connection  $\mathbb{D} : \mathfrak{X}(M_o) \times \Gamma(\mathcal{S}) \longrightarrow \Gamma(\mathcal{S})$ , called *spinor connection*, defined by

$$\mathbb{D}_X s \stackrel{\text{def}}{=} \nabla_{\widehat{X}} \widehat{s}|_{M_o} + \mathbb{A}_X s,$$

where  $\widehat{X}, \widehat{s}$  are fields in  $\Gamma_{\text{loc}}(\mathcal{T}M^{\mathcal{M}})$  such that  $\widehat{X}|_{M_o} = X, \widehat{s}|_{M_o} = s$ ;

Notice that the values  $\widehat{g}|_x, x \in M_o$ , of the graviton are identifiable with  $\mathcal{M}$ -bilinear maps  $\widehat{g}|_x : (T_x M_o)^{\mathcal{M}} \times (T_x M_o)^{\mathcal{M}} \longrightarrow \mathcal{M}$  and not with classical scalar products of the tangent spaces  $T_x M_o$ , as in [32]. However,  $\widehat{g}$  is a bosonic field and its components  $\widehat{g}_{ij} = \widehat{g}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$  in any coordinate frame  $\frac{\partial}{\partial x^i}$  of  $M_o$ , are even-valued  $\mathcal{M}$ -functions, which behave as components of a pseudo-Riemannian metric. Moreover, the tensor field  $\widehat{g}^{\mathbb{R}} = \widehat{g}_{ij}^{\mathbb{R}} dx^i \otimes dx^j$ , determined by the  $\mathbb{R}$ -valued parts  $\widehat{g}_{ij}^{\mathbb{R}}$  of the maps  $\widehat{g}_{ij} : \mathcal{U}_o \subset M_o \longrightarrow \mathcal{M} = \mathbb{R} + W + \Lambda^2 W + \dots$ , is a pseudo-Riemannian metric of signature  $(p, q)$  in the usual sense. Notice also that, as in [32], *the (real part of the) metric connection  $D$  is a metric connection for  $\widehat{g}^{\mathbb{R}}$ .*

### 3.4. The Principle of General Covariance on $\mathcal{M}$ -supermanifolds.

From §2, we know that automorphisms, even tensor fields and Lie derivatives by even vector fields of a  $\mathcal{M}$ -supermanifold behave as diffeomorphisms, tensor fields and Lie derivatives of a smooth manifold  $M$ , endowed with a distinguished submanifold  $M_o$  and a sheaf of  $\mathcal{M}$ -valued functions. Moreover, modulo simple adjustments of signs, their expressions in supercoordinates are identical to those on smooth manifolds. This explains why *the “simple-minded approach”, which (locally) deals with supermanifolds as smooth spaces, with points labeled by bosonic and fermionic coordinates (see e.g. [12], p. 338), brings in fact to correct conclusions.*

Due to this, we may safely claim that the results on supergravities, stated in [32] for smooth manifolds, hold for supergravities on  $\mathcal{M}$ -supermanifolds as well. We can also re-formulate the Principle of General Covariance of that paper in terms of  $\mathcal{M}$ -supermanifolds as follows.



First of all, notice that for a supergravity  $\mathcal{G} = ((M^{\mathcal{M}}, M_o, \mathcal{D}), (g, \nabla))$  of type  $\mathfrak{g}$ , any (local) automorphism  $\varphi : M^{\mathcal{M}} \rightarrow M^{\mathcal{M}}$  determines the new supergravity of type  $\mathfrak{g}$

$$\mathcal{G}' = \varphi_*(\mathcal{G}) \stackrel{\text{def}}{=} ((M^{\mathcal{M}}, M_o^{\mathcal{M}}, \varphi_*(\mathcal{D})), ((\varphi^{-1})^*g, (\varphi^{-1})^*\nabla)) . \quad (3.3)$$

The principle can be now stated as follows.

*A collection  $\mathcal{E}_o$  of constraints and equations on physical fields of supergravities of type  $\mathfrak{g}$  satisfies the Generalized Principle of Infinitesimal General Covariance if:*

- i) *there is a system  $\mathcal{E}$  of constraints and equations on  $(\mathcal{D}, g, \nabla)$ , such that any (local) solution of  $\mathcal{E}$  determines physical fields which solve  $\mathcal{E}_o$ , and every (local) solution of  $\mathcal{E}_o$  can be obtained in this way;*
- ii) *the class of (local) solutions of  $\mathcal{E}_o$  is invariant under all actions (3.3), where  $\mathcal{G}$  is given by a solution of  $\mathcal{E}$  and  $\varphi = \Phi_t^X$  is a flow of an even supervector field  $X \in \Gamma_{\text{loc}}(\mathcal{T}M^{\mathcal{M}})$ .*

*The system  $\mathcal{E}_o$  is called manifestly covariant if there exist a system  $\mathcal{E}$ , which satisfies (i) and is of tensorial type.*

A manifestly covariant system  $\mathcal{E}_o$  automatically satisfies the Generalized Principle of General Covariance ([32]).

#### 4. SUPERGRAVITY IN 11 DIMENSIONS

##### 4.1. Notation.

Let  $\mathfrak{g} = \mathfrak{so}(V) + V + S$  be a super Poincaré algebra with  $(V, <, >) = \mathbb{R}^{p,q}$ ,  $n = p + q$ . We always assume that:

- $(e_A) = (e_i, e_\alpha)$  is a fixed basis for  $V + S$ , with  $(e_i)$  orthonormal basis of  $(V, <, >)$ , i.e.  $< e_i, e_j > = \epsilon_i \delta_{ij}$  with  $\epsilon_i = \begin{cases} 1 & \text{if } 1 \leq i \leq p, \\ -1 & \text{if } p+1 \leq i \leq n; \end{cases}$
- $(e^A) = (e^i, e^\alpha)$  is the dual basis for  $V^* + S^*$  and  $\omega_o = e^1 \wedge \dots \wedge e^n$ ; we use the notation  $\epsilon_{j_1 \dots j_n} \stackrel{\text{def}}{=} \omega_o(e_{j_1}, \dots, e_{j_n})$ ;
- $(\cdot)^\sharp : \bigotimes^r V^* \longrightarrow \bigotimes^r V$  is the isomorphism induced by the duality map  $(\cdot)^\sharp : V^* \longrightarrow V$  determined by the relation  $< \alpha^\sharp, v > = \alpha(v)$ ;
- $* : \Lambda^r V^* \longrightarrow \Lambda^{n-r} V^*$  is the Hodge star operator, determined by  $\omega_o$ , i.e. if  $\alpha = \sum_{j_1 < \dots < j_r} \alpha_{j_1 \dots j_r} e^{j_1} \wedge \dots \wedge e^{j_r}$  and  $\alpha^\sharp = \sum_{m_1 < \dots < m_r} \alpha^{m_1 \dots m_r} e_{m_1} \wedge \dots \wedge e_{m_r}$ , then
 
$$*\alpha = \frac{1}{r!(n-r)!} \epsilon_{m_1 \dots m_r j_1 \dots j_{n-r}} \alpha^{m_1 \dots m_r} e^{j_1} \wedge \dots \wedge e^{j_{n-r}} ;$$
- $(M^{\mathcal{M}}, M_o^{\mathcal{M}}, \mathcal{D})$  is a super-spacetime of type  $\mathfrak{g}$ ;
- $g$  is a tensor field of type  $(0, 2)$  on  $M^{\mathcal{M}}$ , satisfying Definition 3.3 (i) and  $\mathcal{D}^\perp$  is the corresponding complementary distribution;

- $(E_A) = (E_i, E_\alpha)$  is a frame field on an open subset  $\mathcal{U}^{\mathcal{M}}$  of  $M^{\mathcal{M}}$ , associated with  $(e_A)$  and satisfying Definition 3.3 (i);  $(E^A) = (E^i, E^\alpha)$  is the dual coframe field.

For a fixed choice of  $g$ , we denote by  $(\cdot)^\sharp : \bigotimes^r \mathcal{T}^* M^{\mathcal{M}} \longrightarrow \bigotimes^r \mathcal{T} M^{\mathcal{M}}$  the isomorphism induced by the duality

$$(\cdot)^\sharp : \mathcal{T}^* M^{\mathcal{M}} \longrightarrow \mathcal{T} M^{\mathcal{M}} \quad \text{such that} \quad g(\alpha^\sharp, X) = \alpha(X)$$

for any  $X \in \Gamma_{\text{loc}}(\mathcal{T} M)$ ,  $\alpha \in \Gamma_{\text{loc}}(\mathcal{T}^* M)$ . If  $\alpha = \alpha_A E^A$ , then  $\alpha^\sharp = (\eta^{AB} \alpha_B) E_A$  where  $[\eta^{AB}] = [\eta_{AB}]^{-1}$  with  $\eta_{AB} \stackrel{\text{def}}{=} (e_A, e_B)$ .

#### 4.2. Clifford products.

Let  $\mathcal{B} = (e_i)$  be an orthonormal basis for  $V = \mathbb{R}^{p,q}$ , and, given a spinor representation  $\cdot : \mathcal{Cl}(V) \times S \longrightarrow S$  of the Clifford algebra  $\mathcal{Cl}(V) = \mathcal{Cl}_{p,q}$  onto  $S = \mathbb{K}^N$ ,  $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ , let us denote by  $\Gamma_i \in \mathfrak{gl}_N(\mathbb{K})$ ,  $1 \leq i \leq p+q$ , the matrices associated with the  $e_i$ 's, i.e. such that  $(e_i \cdot s)^\alpha = \Gamma_{i\beta}^\alpha s^\beta$  for any  $s = (s^\alpha) \in S = \mathbb{K}^N$ .

Recall that there is a natural *vector space* isomorphism  $\varphi : \mathcal{Cl}(V) \longrightarrow \Lambda^* V$ , which can be used to define the following “Clifford product”

$$\cdot : \Lambda^r V \times S \longrightarrow S, \quad B \cdot s \stackrel{\text{def}}{=} \varphi^{-1}(B) \cdot s.$$

If  $B = \sum_{j_1 < \dots < j_r} B^{j_1 \dots j_r} e_{j_1} \wedge \dots \wedge e_{j_r}$  and  $s = (s^\alpha)$ , the components of  $B \cdot s$  are

$$(B \cdot s)^\alpha \stackrel{\text{def}}{=} \sum_{j_1 < \dots < j_r} B^{j_1 \dots j_r} (\Gamma_{j_1} \cdot \dots \cdot \Gamma_{j_r})_\beta^\alpha s^\beta.$$

Clifford products between fields in  $\mathcal{D}^\perp$  and fields in  $\mathcal{D}$  are defined as follows.

**Definition 4.1.** A tensor field  $w \in \Gamma_{\text{loc}}(\Lambda^r \mathcal{T} M^{\mathcal{M}})$  is called  *$\mathcal{D}$ -orthogonal* if it takes values in the sheaf generated by wedge products of vector fields in  $\mathcal{D}^\perp$ . An  $r$ -form  $\omega \in \Gamma_{\text{loc}}(\Lambda^r \mathcal{T}^* M^{\mathcal{M}})$  is called  *$\mathcal{D}$ -orthogonal* if  $\omega^\sharp$  is  $\mathcal{D}$ -orthogonal. The sheaves of  $\mathcal{D}$ -orthogonal  $r$ -forms and skew-symmetric tensor fields of type  $(r, 0)$  will be denoted by  $\Lambda^r \mathcal{D}^{\perp*}$  and  $\Lambda^r \mathcal{D}^\perp$ , respectively.

Notice that, when  $(E_A) = (E_i, E_\alpha)$  is a frame field of §4.1, a tensor field  $w$  is in  $\Gamma_{\text{loc}}(\Lambda^r \mathcal{D}^\perp)$  if and only if it is the form  $w = \sum_{j_1 < \dots < j_r} w^{j_1 \dots j_r} E_{j_1} \wedge \dots \wedge E_{j_r}$ . Given  $w \in \Gamma_{\text{loc}}(\Lambda^r \mathcal{D}^\perp)$ ,  $s = s^\alpha E_\alpha \in \Gamma_{\text{loc}}(\mathcal{D})$ , we call *Clifford product between  $w$  and  $s$*  the vector field in  $\mathcal{D}$

$$w \cdot s \stackrel{\text{def}}{=} \left( \sum_{j_1 < \dots < j_r} w^{j_1 \dots j_r} (\Gamma_{j_1} \cdot \dots \cdot \Gamma_{j_r})_\beta^\alpha s^\beta \right) E_\alpha.$$

By  $\text{SO}(V)$ -equivariance,  $w \cdot s$  does not depend on the frame field  $(E_A)$ .

### 4.3. Orientations, $\mathcal{D}^\perp$ -curvatures and Rarita-Schwinger 1-form.

**Definition 4.2.** Let  $\mathcal{G} = ((M^{\mathcal{M}}, M_o^{\mathcal{M}}, \mathcal{D}), (g, \nabla))$  be a supergravity of type  $\mathfrak{g}$ . A  $\mathcal{D}^\perp$ -volume form is an even  $n$ -form  $\omega \in \Gamma(\Lambda^n \mathcal{D}^{\perp*})$ , satisfying the following condition: for any system of coordinates  $\xi = (x^i) : \mathcal{U}_o \subset M_o \rightarrow \mathbb{R}^n$  and for any  $x \in \mathcal{U}_o$ , the element in  $\mathcal{M} = \Lambda^* W$

$$\lambda(x) = \omega|_{M_o} \left( \left. \frac{\partial}{\partial x^1} \right|_x, \dots, \left. \frac{\partial}{\partial x^n} \right|_x \right)$$

is invertible, i.e. its component  $\lambda^{\mathbb{R}}(x)$  in  $\mathbb{R} \subset \Lambda^* W$  is different from 0. If there is a  $\mathcal{D}^\perp$ -volume form, we say that  $\mathcal{G}$  is *orientable*. We call it *oriented* when it is endowed with a fixed  $\mathcal{D}^\perp$ -volume form  $\omega$ , determined up multiplication by an invertible  $\mathcal{M}$ -superfunction  $\lambda$ .

Notice that  $\mathcal{G}$  is orientable if and only if  $M_o$  is orientable.

Given a  $\mathcal{D}^\perp$ -volume form  $\omega$ , a frame field  $(E_A)$  as in §4.1 is called *positively oriented* if there is a  $\mathcal{M}$ -superfunction  $\lambda$  such that  $\omega = \lambda (E^1 \wedge \dots \wedge E^n)$  with  $\lambda^{\mathbb{R}}|_x > 0$  for any  $x \in M_o$ . Notice that, for any given  $x_o \in M_o$ , it is always possible to determine a positively oriented frame field  $(E_A)$  on a neighborhood  $\mathcal{U}^{\mathcal{M}} \subset M^{\mathcal{M}}$  of  $x_o$ . In the following, when  $M^{\mathcal{M}}$  is oriented, the frame fields are tacitly assumed to be positively oriented.

Consider now an oriented supergravity  $\mathcal{G} = ((M^{\mathcal{M}}, M_o^{\mathcal{M}}, \mathcal{D}), (g, \nabla))$ . We call *Hodge star operator* the linear operator  $*$  :  $\Gamma_{\text{loc}}(\Lambda^r \mathcal{D}^{\perp*}) \rightarrow \Gamma_{\text{loc}}(\Lambda^{n-r} \mathcal{D}^{\perp*})$  defined as follows. Given a positively oriented frame field  $(E_A)$  and a  $\mathcal{D}$ -orthogonal  $r$ -form  $w$ , we know that  $w^\sharp$  is (locally) of the form  $w^\sharp = \sum_{m_1 < \dots < m_r} w^{m_1 \dots m_r} E_{m_1} \wedge \dots \wedge E_{m_r}$ . We define

$$*w = \sum_{m_1 < \dots < m_r; j_1 < \dots < j_{n-r}} \epsilon_{m_1 \dots m_r j_1 \dots j_{n-r}} w^{m_1 \dots m_r} E^{j_1} \wedge \dots \wedge E^{j_{n-r}}.$$

One can check that  $*w$  is independent of the choice of  $(E_A)$ : it depends only on  $w$ ,  $g$  and the orientation.

Let  $z, z' \in \Gamma_{\text{loc}}(\Lambda^r \mathcal{D}^{\perp*})$ , with components in a frame field

$$z = \sum_{m_1 < \dots < m_r} z_{m_1 \dots m_r} E^{m_1} \wedge \dots \wedge E^{m_r}, \quad z'^\sharp = \sum_{j_1 < \dots < j_r} z'^{j_1 \dots j_r} E_{j_1} \wedge \dots \wedge E_{j_r}.$$

We call *inner product between  $z$  and  $z'$*  the  $\mathcal{M}$ -superfunction  $g(z, z') \stackrel{\text{def}}{=} z_{m_1 \dots m_r} z'^{m_1 \dots m_r}$ . Notice that  $g(z, z')$  is independent of the choice of the frame field. By little abuse of notation, we denote  $\|z\|_g^2 = g(z, z)$ , even if “ $\|z\|_g$ ” does not exist.

In analogy with [32], we denote by  $g^{\mathcal{D}^\perp}$ ,  $\text{Ric}^{\mathcal{D}^\perp}$  and  $s^{\mathcal{D}^\perp}$  the even tensor fields of type  $(0, 2)$  and even  $\mathcal{M}$  superfunction

$$g^{\mathcal{D}^\perp}(X, Y) = g(\pi^{\mathcal{D}^\perp}(X), \pi^{\mathcal{D}^\perp}(Y)),$$

$$\begin{aligned}\mathrm{Ric}^{\mathcal{D}^\perp}(X, Y) &= \sum_{i=1}^n \epsilon_i g(R_{\pi^{\mathcal{D}^\perp}(X)E_i} \pi^{\mathcal{D}^\perp}(Y), E_i) , \\ s^{\mathcal{D}^\perp} &= \sum_{j=1}^n \epsilon_j \mathrm{Ric}^{\mathcal{D}^\perp}(E_j, E_j) ,\end{aligned}$$

where  $(E_A) = (E_i, E_\alpha)$  is a frame field as in §4.1 and  $\epsilon_i = g(E_i, E_i) = \pm 1$ .

Finally, we call *Rarita-Schwinger 1-form* the even tensor field  $\mathcal{R} \in \Gamma(\mathcal{D} \otimes \mathcal{T}^* M^{\mathcal{M}})$  defined by

$$\mathcal{R}(X) \stackrel{\text{def}}{=} \sum_{i < j} \epsilon_i \epsilon_j (\pi^{\mathcal{D}^\perp}(X) \wedge E_i \wedge E_j) \cdot \{(\pi^{\mathcal{D}} \circ T)(E_i, E_j)\}$$

where  $(E_i, E_\alpha)$  is a frame field as in §4.1 and “ $\cdot$ ” denotes a Clifford product.

The tensor fields on  $M_o$  given by the restrictions  $\mathrm{Ric}^{\mathcal{D}^\perp}|_{\sqrt{2}TM_o^{\mathcal{M}}}$  and  $\mathcal{R}|_{TM_o^{\mathcal{M}}}$  can be written in terms of graviton, gravitino, Ricci tensor of the metric connection  $D$  and covariant derivatives  $\mathbb{D}_{\frac{\partial}{\partial x^i}}(\vartheta(\frac{\partial}{\partial x^j}))$ . To check this, we refer to [32] since the required expressions are formally identical to those that one can derive on a non-super space-time.

#### 4.4. Supergravity in 11 dimensions.

According to the remarks in §3.4, *all results, established in [32] for non-super space-times, can be re-formulated in the context of super space-times*, provided that appropriate adjustments in signs are taken into account.

So, as in §4.1 of [32], if  $(g, \nabla)$  is a gravity field on a super-spacetime  $(M^{\mathcal{M}}, M_o^{\mathcal{M}}, \mathcal{D})$ , the torsion  $T$  of  $\nabla$  decomposes into a sum of the form

$$T = T^{\mathcal{D}^\perp} + T_x^{\mathcal{D}} + \mathcal{C}^{\mathcal{D}, \mathcal{D}^\perp; \mathcal{D}} + \mathcal{C}^{\mathcal{D}, \mathcal{D}^\perp; \mathcal{D}^\perp} + \mathcal{H}^{\Lambda^2 \mathcal{D}^\perp; \mathcal{D}} + \mathcal{H}^{\Lambda^2 \mathcal{D}; \mathcal{D}^\perp} ,$$

with

$$\begin{aligned}T^{\mathcal{D}^\perp} &\in \Gamma(\mathcal{D}^\perp \otimes \Lambda^2 \mathcal{D}^{\perp*}) , & T^{\mathcal{D}} &\in \Gamma(\mathcal{D} \otimes \Lambda^2 \mathcal{D}^*) , \\ \mathcal{C}^{\mathcal{D}, \mathcal{D}^\perp; \mathcal{D}} &\in \Gamma(\mathcal{D} \otimes \mathcal{D}^* \otimes \mathcal{D}^{\perp*}) , & \mathcal{C}^{\mathcal{D}, \mathcal{D}^\perp; \mathcal{D}^\perp} &\in \Gamma(\mathcal{D}^\perp \otimes \mathcal{D}^* \otimes \mathcal{D}^{\perp*}) , \\ \mathcal{H}^{\Lambda^2 \mathcal{D}^\perp; \mathcal{D}} &\in \Gamma(\mathcal{D} \otimes \Lambda^2 \mathcal{D}^{\perp*}) , & \mathcal{H}^{\Lambda^2 \mathcal{D}; \mathcal{D}^\perp} &\in \Gamma(\mathcal{D}^\perp \otimes \Lambda^2 \mathcal{D}^*) ,\end{aligned}$$

where  $\mathcal{D}^{\perp*}, \mathcal{D}^* \subset \mathcal{T}^* M^{\mathcal{M}}$  are the sheaves of 1-forms which vanish identically on sections of  $\mathcal{D}$  and  $\mathcal{D}^\perp$ , respectively. Moreover, as in [32],

- $\mathcal{H}^{\Lambda^2 \mathcal{D}; \mathcal{D}^\perp} = -\mathcal{L}$ , where  $\mathcal{L}$  is the Levi tensor  $\mathcal{L}$  of  $\mathcal{D}$ , given by  $\mathcal{D}^\perp$ ;
- for any tensor field  $g$  on  $(M^{\mathcal{M}}, M_o^{\mathcal{M}}, \mathcal{D})$ , satisfying Definition 3.3 (i), there exists an essentially unique connection  $\nabla$  such that

$$T^{\mathcal{D}^\perp} = 0 \quad \text{and} \quad \mathcal{C}^{\mathcal{D}, \mathcal{D}^\perp; \mathcal{D}^\perp} \in \Gamma(\mathrm{Sym}(\mathcal{D}^\perp) \otimes \mathcal{D}^*) , \quad (4.1)$$

where  $\mathrm{Sym}(\mathcal{D}^\perp) \otimes \mathcal{D}^*$  is the sheaf of the sections  $C \in \mathcal{D}^\perp \otimes \mathcal{D}^{\perp*} \otimes \mathcal{D}^*$  satisfying  $g(C(s, V), V') = (-1)^{|s||V|} g(V, C(s, V'))$  for any homogeneous  $V, V' \in \Gamma_{\mathrm{loc}}(\mathcal{D}^\perp)$ ,  $s \in \Gamma_{\mathrm{loc}}(\mathcal{D})$  (for detailed statement, see [32], Thm. 4.1).

Supergravities satisfying (4.1) are called *Levi-Civita*, while we call *strong Levi-Civita* the supergravities satisfying the following stronger constraints:

- 1)  $T^{\mathcal{D}^\perp} = 0 = \mathcal{C}^{\mathcal{D}, \mathcal{D}^\perp; \mathcal{D}^\perp}$  (i. e. *strict Levi-Civita* according to [32]),
- 2)  $T^{\mathcal{D}} \equiv 0$ .

We point out that (1) and (2) are the constraints, which appear in super-space formulation of simple  $4D$ -supergravity (see [35, 34, 26, 32]) and, as we will shortly see, play a crucial role in the theory of supergravities in 11 dimensions.

Let us now show how the theory of 11D-supergravity by Cremmer, Julia and Scherk ([14]) and its (on-shell) superspace formulation ([13, 11] (see also [12]) can be presented in terms of supergravities of type  $\mathfrak{g}$ .

Let  $V = \mathbb{R}^{10,1}$  and  $\mathfrak{g} = \mathfrak{so}(V) + V + S$  the super-Poincaré algebra, determined by the admissible bilinear form  $\beta(s, s') = \text{Im}(is^T \Gamma_0 s')$  on the irreducible module  $S = \mathbb{C}^{32}$  of Dirac spinors, with Dirac matrices  $\Gamma_0$  antisymmetric and  $\Gamma_i$ ,  $i \neq 0$ , symmetric.

**Definition 4.3.** We call *CJS-supergravity* a triple  $\mathcal{G}_{CJS} = ((M^{\mathcal{M}}, M_o^{\mathcal{M}}, \mathcal{D}), (g, \nabla), \mathcal{F})$ , formed by

- an oriented super space-time  $(M^{\mathcal{M}}, M_o^{\mathcal{M}}, \mathcal{D})$  of type  $\mathfrak{g}$ ;
- a gravity field  $(g, \nabla)$  on  $M^{\mathcal{M}}$ ;
- an even  $\mathcal{D}$ -orthogonal, 4-form  $\mathcal{F}$  on  $M^{\mathcal{M}}$ ,

and subjected to the following constraints

- 1)  $\nabla$  is strong Levi-Civita (i.e.  $T^{\mathcal{D}^\perp} = 0 = \mathcal{C}^{\mathcal{D}, \mathcal{D}^\perp; \mathcal{D}^\perp} = T^{\mathcal{D}}$ );
- 2) for any even  $X \in \Gamma_{\text{loc}}(\mathcal{D}^\perp)$  and odd  $s \in \Gamma_{\text{loc}}(\mathcal{D})$

$$\mathcal{C}^{\mathcal{D}, \mathcal{D}^\perp; \mathcal{D}}(s, X) = \frac{1}{144} (X \wedge \mathcal{F}^\sharp - 8(\iota_X \mathcal{F})^\sharp) \cdot s ,$$

where “ $\cdot$ ” denotes Clifford product.

If  $\mathcal{G}_{CJS}$  is a CJS-supergravity, its *super-flux* is the skew-symmetric tensor field  $F \stackrel{\text{def}}{=} \mathcal{F} + \mathcal{Z}$  with  $\mathcal{Z}$  defined by

$$\begin{aligned} \mathcal{Z}(X_1, X_2, X_3, X_4) &\stackrel{\text{def}}{=} \\ &= \frac{1}{8} \sum_{\sigma \in P_4} \varepsilon(\sigma, X) g \left( i\pi^{\mathcal{D}}(X_{\sigma(1)}), \left( \pi^{\mathcal{D}^\perp}(X_{\sigma(2)}) \wedge \pi^{\mathcal{D}^\perp}(X_{\sigma(3)}) \right) \cdot \pi^{\mathcal{D}}(X_{\sigma(4)}) \right) \end{aligned}$$

where  $\varepsilon(\sigma, X)$  is the super-sign (A.3) and “ $\cdot$ ” denotes Clifford product.

Given a CJS-supergravity  $\mathcal{G}_{CJS}$ , we consider as *physical fields* of  $\mathcal{G}_{CJS}$  the graviton, the gravitino, the A-field and the metric and spinor connections, defined as for any other supergravity, plus the *flux* that is the tensor field  $\mathbb{F} \in \Gamma(\Lambda^4 T^* M_o^{\mathcal{M}})$  defined by

$$\mathbb{F}(X_1, X_2, X_3, X_4) \stackrel{\text{def}}{=} F \left( \widehat{X}_1, \widehat{X}_2, \widehat{X}_3, \widehat{X}_4 \right) \Big|_{M_o} ,$$

where the  $\widehat{X}_i \in \Gamma_{\text{loc}}(\mathcal{T}M^{\mathcal{M}})$  are such  $\widehat{X}_i|_{M_o} = X_i$ . Notice that, by (1), (2) and definition of  $F$ , the A-field and the metric and spinor connections are completely determined by  $g$ ,  $\vartheta$  and  $\mathbb{F}$ , so that the degrees of freedom of all physical fields are given only by these three fields.

We claim that *Cremmer, Julia and Scherk's theory can be considered as the theory of CJS-supergravities that solve the system of equations:*

$$\begin{aligned} \text{i)} \quad & \begin{cases} (d\mathcal{F} + d\mathcal{Z})|_{TM_o \times \dots \times TM_o} = 0, \\ \left( (d * \mathcal{F})^{\mathcal{D}^\perp} + \mathcal{F} \wedge \mathcal{F} \right)|_{TM_o \times \dots \times TM_o} = 0 \end{cases} & (\text{Maxwell equations}); \\ \text{ii)} \quad & \mathcal{R}|_{TM_o} = 0 & (\text{Rarita-Schwinger equations}); \\ \text{iii)} \quad & \left( \text{Ric}^{\mathcal{D}^\perp} - \frac{1}{2} s^{\mathcal{D}^\perp} g^{\mathcal{D}^\perp} - \frac{1}{24} \left( \|\mathcal{F}\|_g^2 g^{\mathcal{D}^\perp} - 8g(\iota_{(\cdot)} \mathcal{F}, \iota_{(\cdot)} \mathcal{F}) \right) \right)|_{TM_o \times TM_o} = 0 & (\text{Einstein equations}). \end{aligned}$$

(here  $(\cdot)^{\mathcal{D}^\perp}$  is the projection onto the space of  $\mathcal{D}$ -orthogonal forms). In fact, if  $\widehat{g}$ ,  $\vartheta$ ,  $\mathbb{F}$  are graviton, gravitino and flux of a CJS-supergravity satisfying (i) - (iii), then they satisfy the Euler-Lagrange equations of Cremmer, Julia and Scherk's Lagrangian for 11D-supergravity: just look at expressions in coordinates of (i) - (iii) and compare them with the equations in [13].

The converse, i.e. if  $\widehat{g}$ ,  $\vartheta$ ,  $\mathbb{F}$  satisfy Cremmer, Julia and Scherk's equations, then they are physical fields of a CJS-supergravity satisfying (i) - (iii), can be checked as follows. We give here only an informal sketch, planning to give detailed arguments somewhere else.

For any CJS-supergravity  $\mathcal{G}_{CJS} = ((M^{\mathcal{M}}, M_o^{\mathcal{M}}, \mathcal{D}), (g, \nabla), \mathcal{F})$ , one may consider the  $\text{SO}(V)$ -superbundle  $\pi : P^{\mathcal{M}} \rightarrow M^{\mathcal{M}}$ , generated by orthonormal frame fields as in §4.1 <sup>(2)</sup>. The superbundle  $P^{\mathcal{M}}$  is endowed with the  $V + S$ -valued, soldering 1-form  $\theta = e_C \otimes_{\mathcal{M}} \theta^C = e_i \otimes_{\mathcal{M}} \theta^i + e_\alpha \otimes_{\mathcal{M}} \theta^\alpha$  and the  $\mathfrak{so}(V)$ -valued, connection 1-form  $\omega^\nabla = E_A^B \otimes_{\mathcal{M}} \omega_B^A$ , corresponding to the covariant derivation  $\nabla$ . Here,  $\theta^C$ ,  $\omega_B^A$  are  $\mathcal{M}$ -valued 1-forms,  $(e_C) = (e_i, e_\alpha)$  is a basis for  $V + S$  as in §4.1, and  $(E_A^B)$  is the basis of  $\mathfrak{gl}(V + S)$  with elements defined by  $E_A^B \cdot e_C = \delta_C^B e_A$ ; the 1-forms  $\omega_B^A$  are such that  $\omega^\nabla = E_A^B \otimes_{\mathcal{M}} \omega_B^A$  takes values in  $\mathfrak{so}(V) \subset \mathfrak{gl}(V + S)$ .

As for classical smooth manifolds, the torsion  $\widetilde{T}$  and curvature  $\widetilde{R}$  of  $\omega^\nabla$  are  $\text{SO}(V)$ -equivariant 2-forms on  $P^{\mathcal{M}}$  and induce on  $M^{\mathcal{M}}$  the torsion  $T$  and curvature  $R$  of  $\nabla$ . There exists also a (uniquely defined)  $\text{SO}(V)$ -equivariant 4-form  $\widetilde{\mathcal{F}} = \widetilde{F}_{i_1 i_2 i_3 i_4} \theta^{i_1} \wedge \theta^{i_2} \wedge \theta^{i_3} \wedge \theta^{i_4}$  on  $P^{\mathcal{M}}$ , which induces the 4-form  $\mathcal{F}$  on  $M^{\mathcal{M}}$ . One can check that if constraints (1), (2) and equation  $d\mathcal{F} + d\mathcal{Z} = 0$  are satisfied, then  $\widetilde{T}$ ,  $\widetilde{R}$  and  $\frac{1}{3}\widetilde{\mathcal{F}}$  are the curvatures of a Free Differential Algebra  $\mathcal{A}$  satisfying the constraints given in [12] (III.8.34) - (III.8.37) (for

<sup>2</sup>For brevity, we omit the definitions of “principal superbundles” and related notions, but they can be guessed by analogy from corresponding classical definitions.

definition of Free Differential Algebras and applications to supergravity, see [12, 15, 17]).

It follows that  $\tilde{T}$ ,  $\tilde{R}$  and  $\frac{1}{3}\tilde{\mathcal{F}}$  satisfy a system of equations, given by the generalized Bianchi identities of  $\mathcal{A}$  and the integrability conditions, which are consequences of constraints and Bianchi identities (see [12], p. 908–910). Neglecting the usual Bianchi identities of  $\tilde{T}$  and  $\tilde{R}$  (which are automatically satisfied because we are assuming that  $\omega^\nabla$  is a connection form) and the relations identically satisfied because of the constraints, the system is equivalent to the following set of equations on tensor fields on  $M^\mathcal{M}$ :

$$\begin{aligned} \text{i')} \quad & \begin{cases} dF = d\mathcal{F} + d\mathcal{Z} = 0, \\ (d * \mathcal{F})^{\mathcal{D}^\perp} + \mathcal{F} \wedge \mathcal{F} = 0; \end{cases} \\ \text{ii')} \quad & \mathcal{R} = 0; \\ \text{iii')} \quad & \text{Ric}^{\mathcal{D}^\perp - \frac{1}{2}s^{\mathcal{D}^\perp}} g^{\mathcal{D}^\perp} - \frac{1}{24} \left( \|\mathcal{F}\|_g^2 g^{\mathcal{D}^\perp} - 8g(\iota_{(\cdot)} \mathcal{F}, \iota_{(\cdot)} \mathcal{F}) \right) = 0. \end{aligned}$$

Now, we recall that [12] (III.8.34) - (III.8.37) are “rheonomic constraints” and that any collection of differential forms on  $P_{\|M_o}^\mathcal{M}$ , satisfying restrictions of the above generalized Bianchi identities and integrability conditions, admits a unique extension to a Free Differential Algebra  $\mathcal{A}$  on  $P^\mathcal{M}$  satisfying those constraints (see [12], Ch. III. 3). On the other hand, (i) - (iii) are restrictions to vector fields in  $TM_o$  of the equations (i') - (iii') and hence they correspond to restrictions to vector fields in  $T(P_{\|M_o}^\mathcal{M})$  of the Bianchi identities and integrability conditions of  $\mathcal{A}$ . We infer that the triples  $(\hat{g}, \vartheta, \mathbb{F})$  satisfying Cremmer, Julia and Scherk’s equations (which, we recall, are the equation one gets when writes (i) - (iii) in terms of  $(\hat{g}, \vartheta, \mathbb{F})$ ) are in one-to-one correspondence with the Free Differential Algebras  $\mathcal{A}$  on  $P^\mathcal{M}$ , satisfying the quoted rheonomic constraints. Since such Free Differential Algebras correspond uniquely to CJS-supergravities satisfying (i') - (iii'), the claim follows.

It is important to observe that such arguments show also that a CJS-supergravity satisfies (i) - (iii) if and only if it satisfies (i') - (iii'). *Since (1), (2) and (i') - (iii') are of tensorial type, they are manifestly covariant and hence they satisfy the Generalized Principle of General Covariance.*

Finally, as pointed out in [12], we remark that all equations (i') - (iii'), except  $dF = 0$ , are integrability conditions that are automatically satisfied by any CJS-supergravity solving  $dF = 0$ . This intriguing fact was first proved by Brink and Howe and Cremmer and Ferrara in [11, 13].

## APPENDIX A. BASICS OF SUPERGEOMETRY

### A.1. A digest of supermanifolds.

A.1.1. *First definitions.* A *smooth supermanifold of dimension  $(n|m)$*  is a pair  $M = (M_o, \mathcal{A}_M)$ , formed by an  $n$ -dimensional smooth manifold  $M_o$  (called *body*) and a sheaf of  $\mathbb{Z}_2$ -graded algebras  $\pi : \mathcal{A}_M = \mathcal{A}_{M0} + \mathcal{A}_{M1} \rightarrow M_o$  (called *sheaf of superfunctions*), such that

- there exists an open covering  $\{\mathcal{U}_{oJ}\}$  of  $M_o$  with the property that for any restriction  $\mathcal{A}_{M|\mathcal{U}_{oJ}}$  there exists a trivial vector bundle  $\pi : S_J \rightarrow \mathcal{U}_{oJ}$  of rank  $m$  such that  $\mathcal{A}_{M|\mathcal{U}_{oJ}} \simeq \text{Sheaf}(\Lambda S_J^*)$ .
- the sheaf  $\pi : \mathcal{A}_M / (\mathcal{A}_{M1} + \mathcal{A}_{M1}^2) \rightarrow M_o$  is isomorphic with the sheaf  $\mathfrak{F}_{M_o}$  of germs of smooth real functions on  $M_o$ .

Any pair  $\mathcal{U} = (\mathcal{U}_o, \mathcal{A}_{M|\mathcal{U}_o})$ , with an isomorphism  $\mathcal{A}_{M|\mathcal{U}_o} \simeq \text{Sheaf}(\Lambda S^*)$  for a trivial vector bundle  $\pi : S \rightarrow \mathcal{U}_o$ , is called *decomposable neighborhood of  $M$* .

Let  $\mathcal{U} = (\mathcal{U}_o, \mathcal{A}_{M|\mathcal{U}_o})$  be a decomposable neighborhood and assume that there exists coordinates  $\xi = (x^i) : \mathcal{U}_o \rightarrow \mathcal{U}'_o = \xi(\mathcal{U}) \subset \mathbb{R}^n$ , which we use to make the identifications  $\mathcal{U}_o \simeq \mathcal{U}'_o \subset \mathbb{R}^n$  and  $S \simeq \mathbb{R}^m \times \mathcal{U}'_o$ . The corresponding isomorphism  $\widehat{\xi} : \text{Sheaf}(\Lambda \mathbb{R}^{m*} \times \mathcal{U}'_o) \rightarrow \mathcal{A}_{M|\mathcal{U}_o}$  will be called *system of supercoordinates on  $\mathcal{U}$  associated with  $\xi = (x^i)$* .

Denoting by  $(e^\alpha)$  the standard basis of  $\mathbb{R}^{m*}$  and by  $\vartheta^\alpha : \mathcal{U}'_o \rightarrow (\mathbb{R}^m)^* \times \mathcal{U}'_o$  the constant sections  $\vartheta^\alpha(x) \equiv e^\alpha$ , the superfunctions  $\mathfrak{f} \in \Gamma(\mathcal{A}_{M|\mathcal{U}_o})$  can be identified with the sections of  $\text{Sheaf}(\Lambda \mathbb{R}^{m*} \times \mathcal{U}'_o)$

$$\mathfrak{f} = \sum_{\alpha_j=0,1} \mathfrak{f}_{\alpha_1 \dots \alpha_m}(x^1, \dots, x^n) (\vartheta^1)^{\alpha_1} \wedge \dots \wedge (\vartheta^m)^{\alpha_m}, \quad (\text{A.1})$$

where  $(\vartheta^\beta)^0 = 1$  and  $1 \wedge \vartheta^\beta = \vartheta^\beta$ . For simplicity, we set  $(\vartheta^\beta)^\alpha = 0$  for any  $\alpha \neq 0, 1$ , so that (A.1) makes sense even if we sum over  $\alpha_j \in \mathbb{N}$ .

The homogeneous superfunctions  $x^i$  and  $\vartheta^\alpha$  are called *even* and *odd coordinates*, respectively. One can check that even (resp. odd) superfunctions, i.e. superfunctions of parity 0 (resp. 1), are of the form

$$\mathfrak{f} = \sum_{\sum_j \alpha_j = 0 \bmod 2} \mathfrak{f}_{\alpha_1 \dots \alpha_m}(x^1, \dots, x^n) (\vartheta^1)^{\alpha_1} \wedge \dots \wedge (\vartheta^m)^{\alpha_m},$$

$$\left( \mathfrak{f} = \sum_{\sum_j \alpha_j = 1 \bmod 2} \mathfrak{f}_{\alpha_1 \dots \alpha_m}(x^1, \dots, x^n) (\vartheta^1)^{\alpha_1} \wedge \dots \wedge (\vartheta^m)^{\alpha_m} \right).$$

A *morphism* between supermanifolds  $M = (M_o, \mathcal{A}_M)$  and  $N = (N_o, \mathcal{A}_N)$  is a pair  $(f, \widehat{f})$  formed by a smooth map  $f : M_o \rightarrow N_o$  and a morphism  $\widehat{f} : \mathcal{A}_N \rightarrow f_*(\mathcal{A}_M)$  of sheaves of  $\mathbb{Z}_2$ -graded algebras over  $N_o$ .

For any supermanifold  $M = (M_o, \mathcal{A}_M)$ , one can check that the subsheaf  $\mathfrak{J}_M \subset \mathcal{A}_M$  of nilpotent superfunctions, which is generated by germs of the form  $\mathfrak{f} = \sum_{\alpha_1 + \dots + \alpha_m \geq 1} \mathfrak{f}_{\alpha_1 \dots \alpha_m}(x) (\vartheta^1)^{\alpha_1} \wedge \dots \wedge (\vartheta^m)^{\alpha_m}$ , coincides with the sheaf  $(\mathcal{A}_{M1} + \mathcal{A}_{M1}^2)$  and hence that  $\mathcal{A}_M / \mathfrak{J}_M$  is identifiable with  $\mathfrak{F}_{M_o}$ .



The natural projection  $\epsilon : \mathcal{A}_M \longrightarrow \mathcal{A}_M/\mathfrak{I}_M \simeq \mathfrak{F}_{M_o}$  is called *evaluation map*. For a superfunction of the form (A.1),

$$\epsilon(\mathfrak{f}) = \mathfrak{f}_{0\dots 0}(x^1, \dots, x^n) .$$

This fact is often described saying that “ $\epsilon(\mathfrak{f})$  is the function given by evaluating (A.1) at  $\mathcal{U}_o = \{ \vartheta^\alpha = 0 \}$ ”. According to this, for any  $\mathfrak{f} \in \Gamma_{\text{loc}}(\mathcal{A}_M)$  and  $x \in M_o$ , we adopt the notation

$$\mathfrak{f}|_{M_o} = \epsilon(\mathfrak{f}) \quad \text{and} \quad \mathfrak{f}|_x \text{ or } \mathfrak{f}(x) = \epsilon(\mathfrak{f})(x) .$$

The morphism  $\iota_{M_o} = (Id_{M_o}, (\cdot)|_{M_o}) : (M_o, \mathfrak{F}_{M_o}) \longrightarrow M = (M_o, \mathcal{A}_M)$  is called *natural embedding of  $M_o$  into  $M$* .

A supermanifold of dimension  $(0|0)$  and connected body is called *superpoint*. It is unique up to isomorphism and is denoted by  $\mathbb{R}^{0|0}$ . Any  $x \in M_o$  is naturally identified with the superpoint  $(\{x\}, \mathbb{R}) \simeq \mathbb{R}^{0|0}$ . The *natural embedding of  $x$  in  $M$*  is the morphism  $\iota_x = (Id_x, (\cdot)|_x) : (\{x\}, \mathbb{R}) \longrightarrow M$ .

**A.1.2. Cartesian products of supermanifolds.** Let  $M_i = (M_{io}, \mathcal{A}_{M_i})$ ,  $i = 1, 2$ , be two supermanifolds and  $\pi_i : M_{1o} \times M_{2o} \longrightarrow M_{io}$  the natural projection of  $M_{1o} \times M_{2o}$  onto the  $i$ -th factor. The *Cartesian product of  $M_1$  and  $M_2$*  is the supermanifold given by the pair

$$M_1 \times M_2 \stackrel{\text{def}}{=} (M_{o1} \times M_{o2}, \mathcal{A}_{M_1 \times M_2}) ,$$

where  $\pi : \mathcal{A}_{M_1 \times M_2} \longrightarrow M_{o1} \times M_{o2}$  is a sheaf, which is canonically determined by  $\mathcal{A}_{M_1}$ ,  $\mathcal{A}_{M_2}$  and includes  $\pi_1^*(\mathcal{A}_{M_1}) \otimes \pi_2^*(\mathcal{A}_{M_2})$  as a dense subsheaf (for a detailed definition of  $\mathcal{A}_{M_1 \times M_2}$ , see [20], p. 215).

For any  $x \in M_{o1}$ , the *evaluation of superfunctions of  $M_1 \times M_2$  at  $x$*  is the sheaf morphism  $\epsilon_x : \mathcal{A}_{M_1 \times M_2} \longrightarrow \mathcal{A}_{M_2}$  defined by

$$\epsilon_x(\mathfrak{a} \otimes \mathfrak{b}) = \mathfrak{a}|_x \mathfrak{b} \quad \text{for any } \mathfrak{a} \in \Gamma_{\text{loc}}(\mathcal{A}_{M_1}) , \quad \mathfrak{b} \in \Gamma_{\text{loc}}(\mathcal{A}_{M_2}) .$$

**A.1.3. Tensor fields.** The *supervector fields* of a supermanifold  $M = (M_o, \mathcal{A}_M)$  (shortly called *vector fields*) are the derivations of  $\Gamma(\mathcal{A}_M)$ . In supercoordinates, they correspond to derivations of  $\Gamma(\mathcal{A}_{M|\mathcal{U}_o}) = \Gamma(\text{Sheaf}(\Lambda \mathbb{R}^{m*} \times \mathcal{U}'_o))$  of the form

$$X = X^j \frac{\partial}{\partial x^j} + X^\alpha \frac{\partial}{\partial \vartheta^\alpha} , \quad X^j , X^\alpha \in \Gamma(\mathcal{A}_{M|\mathcal{U}_o}) ,$$

where  $\frac{\partial}{\partial x^j}$ ,  $\frac{\partial}{\partial \vartheta^\alpha}$  are such that  $\frac{\partial}{\partial x^j} x^k = \delta_j^k$ ,  $\frac{\partial}{\partial \vartheta^\alpha} \vartheta^\beta = \delta_\alpha^\beta$ ,  $\frac{\partial}{\partial x^j} \vartheta^\alpha = \frac{\partial}{\partial \vartheta^\alpha} x^k = 0$ .

The sheaf  $\pi : \mathcal{T}M \longrightarrow M_o$  of germs of vector fields is called *tangent sheaf*. It has a natural structure of sheaf of  $\mathbb{Z}_2$ -graded  $\mathcal{A}_M$ -modules. The vector fields  $\frac{\partial}{\partial x^j}$ ,  $\frac{\partial}{\partial \vartheta^\alpha}$  have parity 0 and 1, respectively.

The *Lie bracket* of homogeneous  $X, Y \in \Gamma(\mathcal{T}M)$  is defined by

$$[X, Y] \cdot f \stackrel{\text{def}}{=} X \cdot (Y \cdot f) - (-1)^{|X||Y|} Y \cdot (X \cdot f) , \quad \text{for any } f \in \Gamma_{\text{loc}}(\mathcal{A}_M) . \quad (\text{A.2})$$

This operation is extended  $\mathbb{R}$ -bilinearly on arbitrary pairs  $X, Y \in \Gamma(\mathcal{T}M)$ .

For any  $x \in M_o$ , the *tangent space of  $M$  at  $x$*  is the  $\mathbb{Z}_2$ -graded vector space  $T_x M = (T_x M)_0 + (T_x M)_1$ , with  $(T_x M)_\alpha$  defined by

$$(T_x M)_\alpha \stackrel{\text{def}}{=} \{ v : \mathcal{A}_M|_x \rightarrow \mathbb{R} : v(\mathfrak{f}\mathfrak{g}) = v(\mathfrak{f})\mathfrak{g}|_x + (-1)^{|\mathfrak{f}||\mathfrak{g}|} \mathfrak{f}|_x v(\mathfrak{g}) \quad \text{and} \\ v(\mathfrak{f}) = 0 \text{ for any } \mathfrak{f} \in \mathcal{A}_M[\alpha+1]_{\text{mod } 2} \} ,$$

We denote the bundle  $\pi : \bigcup_{x \in M_o} T_x M \longrightarrow M_o$  by “ $TM|_{M_o}$ ”<sup>(3)</sup>.

It is known (see e.g. [20], §2.12) that  $\pi : \text{Sheaf}(TM|_{M_o}) \longrightarrow M_o$  is isomorphic to the sheaf determined by the pre-sheaf

$$\{ \mathcal{U}_o \longrightarrow \text{Der}(\Gamma(\mathcal{A}_M|_{\mathcal{U}_o}), \mathcal{C}_{M_o}^\infty(\mathcal{U}_o)) \} .$$

On the base of such isomorphism, the evaluation map  $\epsilon : \mathcal{A}_M \rightarrow \mathfrak{F}_{M_o}$  determines a surjective map  $\pi^\epsilon : \mathcal{T}M \longrightarrow \text{Sheaf}(TM|_{M_o})$  defined by

$$\pi^\epsilon(X) \cdot \mathfrak{f} \stackrel{\text{def}}{=} \epsilon(X \cdot \mathfrak{f}) .$$

If we consider supercoordinates  $(x^i, \vartheta^\alpha)$  on a decomposable neighborhood  $(\mathcal{U}_o, \mathcal{A}_M|_{\mathcal{U}_o})$  and set  $\frac{\partial}{\partial x^i}|_{M_o} \stackrel{\text{def}}{=} \pi^\epsilon\left(\frac{\partial}{\partial x^i}\right)$ ,  $\frac{\partial}{\partial \vartheta^\alpha}|_{M_o} \stackrel{\text{def}}{=} \pi^\epsilon\left(\frac{\partial}{\partial \vartheta^\alpha}\right)$ , we have that

$$\pi^\epsilon\left(X^j \frac{\partial}{\partial x^j} + X^\alpha \frac{\partial}{\partial \vartheta^\alpha}\right) = X^j|_{\mathcal{U}_o} \frac{\partial}{\partial x^j} \Big|_{M_o} + X^\alpha|_{\mathcal{U}_o} \frac{\partial}{\partial \vartheta^\alpha} \Big|_{M_o} .$$

For any vector field  $X \in \Gamma(\mathcal{T}M)$ , we use the notation

$$X|_{M_o} \stackrel{\text{def}}{=} \pi^\epsilon(X) , \quad X|_x \stackrel{\text{def}}{=} \pi^\epsilon(X)|_x \in T_x M , \quad x \in M_o ,$$

and we say that  $X$  is *tangent to  $M_o$*  if for any  $x \in M_o$  and any system of super-coordinates, one has that  $X|_x$  is of the form  $X|_x = X^i \frac{\partial}{\partial x^i}|_x$  for  $X^i \in \mathbb{R}$ .

For any morphism  $\varphi = (f, \widehat{f}) : M = (M_o, \mathcal{A}_M) \longrightarrow N = (N, \mathcal{A}_N)$ , we denote by  $\varphi_* : f_* \mathcal{T}M \longrightarrow \text{Der}_{\varphi}(\mathcal{A}_N, f_* \mathcal{A}_M)$  the sheaf morphism defined by

$$\varphi_*(X) \cdot \mathfrak{f} \stackrel{\text{def}}{=} X \cdot (\widehat{f}(\mathfrak{f})) \text{ for any } \mathfrak{f} \in \Gamma_{\text{loc}}(\mathcal{A}_N) , \quad X \in \Gamma_{\text{loc}}(f_* \mathcal{T}M) .$$

We conclude with the definition of tensor fields. The *cotangent sheaf of  $M$*  is the sheaf

$$\pi : \mathcal{T}^* M \stackrel{\text{def}}{=} \text{Hom}_{\mathcal{A}_M}(\mathcal{T}M, \mathcal{A}_M) \longrightarrow M_o .$$

A section  $\omega$  of  $\mathcal{T}^* M$  is called *1-form*. It is *homogeneous of parity  $|\omega| \in \mathbb{Z}_2$*  if

$$\omega(\mathcal{T}M_i) \subseteq \mathcal{A}_{M_i+|\omega|} \quad \text{and} \quad \omega(\mathfrak{f}X) = (-1)^{|\omega||\mathfrak{f}|} \mathfrak{f}\omega(X)$$

for any homogeneous  $\mathfrak{f} \in \Gamma_{\text{loc}}(\mathcal{A}_M)$  and  $X \in \Gamma_{\text{loc}}(\mathcal{T}M)$ .

The *full tensor sheaf of  $M$*  is the sheaf

$$\pi : \otimes_{\mathcal{A}_M} \mathcal{T}M, \mathcal{T}^* M \longrightarrow M_o ,$$

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<sup>3</sup>In the literature,  $TM|_{M_o}$  is usually denoted by “ $TM$ ”. We decided to use such new notation, because  $TM|_{M_o}$  is similar more to the restriction of a tangent bundle to a submanifold than to the tangent bundle of a manifold.

generated by the tensor products ( $\mathbb{Z}_2$ -graded over  $\mathcal{A}_M$ ) of  $\mathcal{T}M$  and  $\mathcal{T}^*M$ . A local section  $\alpha$  of  $\otimes_{\mathcal{A}_M} \mathcal{T}M, \mathcal{T}^*M$  is called *tensor field*. It is of *type*  $(p, q)$  if it is sum of tensor products of  $p$  vector fields and  $q$  1-forms. It is called *homogeneous of parity*  $|\alpha| \in \mathbb{Z}_2$  if it is sum of tensor products of homogeneous vector fields and homogeneous 1-forms, whose sum (in  $\mathbb{Z}_2$ ) of parities is equal to  $|\alpha|$ .

**A.1.4. Skew-symmetric tensor fields, exterior differentials and interior multiplications.** A tensor field  $\omega$  of type  $(0, q)$  of  $M$  is called *symmetric* (resp. *skew-symmetric*) *in graded sense* if for any  $q$ -tuple  $X_1, \dots, X_q$  of homogeneous vector fields and any  $1 \leq i \leq q - 1$

$$\begin{aligned} \omega(X_1, \dots, X_i, X_{i+1}, \dots, X_q) &= (-1)^{|X_i||X_{i+1}|} \omega(X_1, \dots, X_{i+1}, X_i, \dots, X_q) \\ &\left( \text{resp.} = -(-1)^{|X_i||X_{i+1}|} \omega(X_1, \dots, X_{i+1}, X_i, \dots, X_q) \right). \end{aligned}$$

Skew-symmetric  $(0, q)$ -tensor fields are also called *q-forms*. Similar definitions are given for symmetric and skew-symmetric  $(p, 0)$ -tensor fields in graded sense. For brevity, the words “in graded sense” are often omitted.

The sheaves over  $M_o$ , generated by skew-symmetric tensor fields of type  $(p, 0)$  and  $(0, q)$ , are denoted by  $\Lambda^p \mathcal{T}M$  and  $\Lambda^q \mathcal{T}^*M$ , respectively, and

$$\Lambda^* \mathcal{T}M \stackrel{\text{def}}{=} \bigoplus_0^\infty \Lambda^p \mathcal{T}M, \quad \Lambda^* \mathcal{T}^*M \stackrel{\text{def}}{=} \bigoplus_0^\infty \Lambda^p \mathcal{T}^*M.$$

For an open subset  $\mathcal{U}_o \subset M_o$ , the space  $\Gamma(\Lambda^* \mathcal{T}^*M|_{\mathcal{U}_o})$  is endowed with a “wedge product”

$$\wedge : \Gamma(\Lambda^q \mathcal{T}^*M|_{\mathcal{U}_o}) \times \Gamma(\Lambda^{q'} \mathcal{T}^*M|_{\mathcal{U}_o}) \longrightarrow \Gamma(\Lambda^{q+q'} \mathcal{T}^*M|_{\mathcal{U}_o}),$$

which, in the context of  $\mathbb{Z}_2$ -graded multilinear maps on  $\mathbb{Z}_2$ -graded vector spaces, is the analogue of wedge products on classical smooth manifolds: differences in the expressions concerns only signs, which have to be consistent with grades of arguments and maps. We refer to [20], p. 244–246, for a detailed definition of “ $\wedge$ ”. We point out that such wedge products determines a natural structure on  $\Lambda^* \mathcal{T}^*M$  of sheaf of bi-graded commutative algebras.

A corresponding definition determines a “wedge product” between sections in  $\Lambda^* \mathcal{T}M|_{\mathcal{U}_o}$  and determines a natural structure of sheaf of bi-graded commutative algebras on  $\Lambda^* \mathcal{T}M$ .

For any homogeneous superfunction  $f \in \Gamma_{\text{loc}}(\mathcal{A}_M)$ , the differential  $df$  is the 1-form, defined (in analogy with the classical case) by

$$df(X) \stackrel{\text{def}}{=} (-1)^{|X||f|} X \cdot f \quad \text{for any homogeneous } X \in \Gamma_{\text{loc}}(\mathcal{T}M).$$

It can be checked ([20], p.249–250) that, for any open subset  $\mathcal{U}_o \subset M_o$ , there exists a unique derivation  $d$  on  $\Gamma(\Lambda^* \mathcal{T}^*M|_{\mathcal{U}_o})$  of bidegree  $(1, 0)$  such that: a)

it coincides with the differential, when applied to homogeneous superfunctions; b) it satisfies  $d^2 = 0$ . Such derivation is called *exterior differential* and it is the analogue of the exterior differential of smooth manifolds. See [20] for its explicit expression and main properties.

For any open subset  $\mathcal{U}_o \subset M_o$  and  $X \in \Gamma(\mathcal{T}M|_{\mathcal{U}_o})$  we denote by  $\iota_X$  the *interior multiplication by  $X$* , i.e. the derivation of  $\Gamma(\Lambda^*\mathcal{T}^*M|_{\mathcal{U}_o})$  of bidegree  $(-1, |X|)$  defined by

$$\iota_X \omega(Y_1, \dots, Y_p) \stackrel{\text{def}}{=} \omega(X, Y_1, \dots, Y_p) .$$

We conclude recalling the definition of “super-signs” of permutations of  $m$  elements, frequently used in constructions of skew-symmetric tensors. For any  $\sigma \in P_m$ , we set  $\Delta_\sigma = \{ (i, j) : 1 \leq i < j \leq m, \sigma(i) > \sigma(j) \}$  and, if  $X = (X_1, \dots, X_m)$  is an  $m$ -tuple of homogeneous vector fields  $X_i \in \Gamma_{\text{loc}}(\mathcal{T}M)$ , we call *super-sign of the pair  $(\sigma, X)$*  the value

$$\varepsilon(\sigma, X) \stackrel{\text{def}}{=} (-1)^{\sum_{(i,j) \in \Delta_\sigma} (1+|X_i||X_j|)} . \quad (\text{A.3})$$

When all the  $X_i$  are even, the super-sign coincides to the classical sign  $\varepsilon(\sigma)$ .

### A.2. Lie supergroups and Lie superalgebras.

Given a supermanifold  $M = (M_o, \mathcal{A}_M)$ , we denote by  $\Delta_M = (\Delta_{M_o}, \hat{\Delta}_M) : M \longrightarrow M \times M$  the *diagonal morphism*, determined by

$$\Delta_{M_o}(x) = (x, x) , \quad \hat{\Delta}_M \circ \hat{\pi}_i(\mathfrak{f}) = \mathfrak{f}$$

where  $\pi_i : M \times M \rightarrow M$ ,  $i = 1, 2$ , are the natural projections.

A *Lie supergroup* is a supermanifold  $G = (G_o, \mathcal{A}_G)$ , with body given by a Lie group  $G_o$  (whose multiplication map is denoted by  $m : G_o \times G_o \longrightarrow G_o$ , inversion map by  $n : G_o \longrightarrow G_o$  and identity by  $e \in G_o$ ) and endowed with morphisms  $\mu = (m, \hat{m}) : G \times G \longrightarrow G$  and  $\nu = (n, \hat{n}) : G \longrightarrow G$ , satisfying the following properties:

- 1) (*associativity*) as morphisms from  $G \times G \times G$  to  $G$

$$\mu \circ (Id_G \times \mu) = \mu \circ (\mu \times Id_G) ;$$

- 2) (*existence of neutral element*) setting  $\mathbf{e} = (\{e\}, \mathbb{R})$ , the following equalities of morphisms, from  $G \times \mathbf{e}$  to  $G$  and from  $\mathbf{e} \times G$  to  $G$ , hold:

$$\mu \circ (Id_G \times \iota_e) = \pi_1 , \quad \mu \circ (\iota_e \times Id_G) = \pi_2 ;$$

- 3) (*inverse elements*) as morphism from  $G$  into  $G$

$$\mu \circ (Id_G \times \nu) \circ \Delta_G = \mu \circ (\nu \times Id_G) \circ \Delta_G = (e, \{f \longmapsto f(e)\}) ,$$

where  $e : G_o \rightarrow G_o$  denotes the constant map with value  $e \in G_o$ .

A *Lie sub-supergroup* of  $G = (G_o, \mathcal{A}_G)$  is a submanifold  $H = (H_o, \mathcal{A}_H)$  of  $G$  (i.e. a supermanifold endowed with an embedding  $\iota = (\iota_o, \hat{\iota}) : H \longrightarrow G$ ), whose body is given by a Lie subgroup  $\iota_o : H_o \rightarrow G_o$  of  $G_o$  and such that  $\mu' = \mu \circ (\iota \times \iota)$  and  $\nu' = \nu \circ \iota$  determine a structure of Lie supergroup on  $H$ .

For any vector field  $X \in \Gamma(\mathcal{T}G)$  of a Lie supergroup of  $G = (G_o, \mathcal{A}_G)$ , let us denote by  $Id \otimes X$  the corresponding derivation of  $\Gamma(\mathcal{T}(G \times G))$  that acts only on the second component. The field  $X$  is called *left-invariant* if it satisfies the condition  $(Id \otimes X) \circ \widehat{m} = \widehat{m} \circ X$  <sup>(4)</sup>. The space of left-invariant vector fields, endowed with the brackets (A.2), is a Lie superalgebra, called the *Lie superalgebra of  $G$* .

**Definition A.1.** A *super Harish-Chandra pair* (shortly *sHC-pair*) is a pair  $(G_o, \mathfrak{g})$ , formed by a Lie group  $G_o$  and a Lie superalgebra  $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$  with  $\mathfrak{g}_0 = Lie(G_o)$ , endowed with a Lie group morphism  $Ad : G_o \rightarrow Aut(\mathfrak{g})$  such that

- $Ad(\cdot)|_{\mathfrak{g}_0} : G_o \rightarrow Aut(\mathfrak{g}_0)$  is the usual adjoint action of  $G_o$ ;
- $Ad_* : \mathfrak{g}_0 \rightarrow aut(\mathfrak{g})$  coincides with  $ad|_{\mathfrak{g}_0} : \mathfrak{g}_0 \rightarrow aut(\mathfrak{g})$ .

Given a Lie supergroup  $G = (G_o, \mathcal{A}_G)$ , one can naturally associate to it the sHC-pair  $(G_o, \mathfrak{g} = Lie(G))$  and *any sHC-pair corresponds to a unique (up to isomorphism) Lie supergroup*. In particular, given a sHC pair  $(G_o, \mathfrak{g})$ , the sheaf  $\mathcal{A}_G$  of the corresponding Lie supergroup  $G = (G_o, \mathcal{A}_G)$  is the one determined by the pre-sheaf on  $G_o$

$$\{ \mathcal{U}_o \rightarrow Hom(U(\mathfrak{g}), \mathcal{C}_{G_o}^\infty(\mathcal{U}_o))^{U(\mathfrak{g}_0)} \} ,$$

where  $U(\mathfrak{g}_0)$  and  $U(\mathfrak{g})$  denote the universal enveloping algebras of  $\mathfrak{g}_0$  and  $\mathfrak{g}$ , respectively. We refer to [21] for the explicit expressions of the product and inverse morphisms  $\mu, \nu$  of  $G = (G_o, \mathcal{A}_G)$ .

Let  $H = (H_o, \mathcal{A}_H)$  be a Lie sub-supergroup of  $G = (G_o, \mathcal{A}_G)$  with  $H_o$  closed in  $G_o$ . Denote by  $p_o : G_o \rightarrow G_o/H_o$ ,

$$p = (p_o, \widehat{p}) : G \times H \rightarrow G \quad \text{and} \quad \mu_H = (m|_{G_o \times H_o}, \widehat{m}_H) : G \times H \rightarrow G$$

the canonical projections and the restriction of the product rule  $\mu = (m, \widehat{m}) : G \times G \rightarrow G$  to  $G \times H$ . The sheaf  $\mathcal{A}_{G/H}$  on  $G_o/H_o$ , determined by the pre-sheaf

$$\left\{ \mathcal{U}_o \rightarrow \left\{ f \in \mathcal{A}_{G \parallel \pi_o^{-1}(\mathcal{U}_o)} : \widehat{m}_H(f) = \widehat{p}(f) \right\} \right\} ,$$

is such that  $G/H = (G_o/H_o, \mathcal{A}_{G/H})$  is a supermanifold, called *homogeneous supermanifold of  $G$  modulo  $H$* . The supergroup  $H$  is called *isotropy of  $G/H$* .

## REFERENCES

- [1] D. V. Alekseevsky and V. Cortés, *Classification of  $N$ -(super)-extended Poincaré algebras and bilinear invariants of the Spinor representation of  $Spin(p, q)$* , Comm. Math. Phys. **183** (1997), 477–510.
- [2] D. V. Alekseevsky, V. Cortés, C. Devchand, U. Semmelmann, *Killing spinors are Killing vector fields in Riemannian supergeometry*, J. Geom. Phys. **26** (1998), 37–50.
- [3] C. Bartocci, U. Bruzzo, D. Hernández-Ruipérez, *The Geometry of Supermanifolds*, Kluwer Academic, Dordrecht, 1991.

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<sup>4</sup>This definition reduces to the usual definition of “left-invariant vector fields” when  $G = G_o$  is a (non-super) Lie group.

- [4] M. Batchelor, *The structure of supermanifolds*, Trans. Amer. Math. Soc. **253** (1979), 329–338.
- [5] M. Batchelor, *Two approaches to supermanifolds*, Trans. Amer. Math. Soc. **258** (1980), 257–270.
- [6] F. A. Berezin, *The method of second quantization*, Academic Press, New York-London, 1966.
- [7] F. A. Berezin and G. I. Kac, *Lie groups with commuting and anticommuting parameters*, Math. USSR-Sb. **11** (1970), 311–326.
- [8] F. A. Berezin and D. A. Leites, *Supermanifolds*, Dokl. Akad. Nauk SSSR **224** (1975), 505–508; English transl. in Soviet Math. Dokl. **16** (1975), 1218–1222 (1976).
- [9] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields*, McGraw-Hill, New York, 1965.
- [10] H. Blaine Lawson, jr. and M.-L. Michelson, *Spin Geometry*, Princeton Univ. Press, Princeton 1989.
- [11] L. Brink and P. Howe, *Eleven-dimensional supergravity on the mass shell in superspace*, Phys. Lett. B **91** (1980), 384–386.
- [12] L. Castellani, R. D’Auria and P. Fré, *Supergravity and Superstrings: A Geometric Perspective - Vol. I-III*, World Scientific, Teaneck, 1991.
- [13] E. Cremmer and S. Ferrara, *Formulation of 11-dimensional supergravity in superspace*, Phys. Lett. B **91** (1980), 61–66.
- [14] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in 11 dimension*, Phys. Lett. B **76** (1978), 409–412.
- [15] R. D’Auria and P. Fré, *Geometric supergravity in  $D = 11$  and its hidden supergroup*, Nucl. Phys. B **201** (1982), 101–140.
- [16] B. DeWitt, *Supermanifolds*, Cambridge Univ. Press, Cambridge, 1984.
- [17] P. Fré, *Comments on the 6-index photon in  $D = 11$  supergravity and the gauging of Free Differential Algebras*, Class. Quantum Grav. **1** (1981), L81.
- [18] D. S. Freed, *Five Lectures on Supersymmetry*, American Mathematical Society, Providence, 1999.
- [19] F. Hélein, *An introduction to supermanifolds and supersymmetry*, preprint downloadable from <http://www.math.jussieu.fr/~helein/articles/supeyr.pdf>, 2008.
- [20] B. Kostant, *Graded manifolds, graded Lie theory and prequantization*, in “Differential Geometric Methods in Mathematical Physics”, Bonn 1975, p. 177–306; Lecture Notes in Mathematics **570**, Springer, Berlin 1977.
- [21] J. L. Koszul, *Graded manifolds and graded Lie algebras* in “Proceedings of the International Meeting on Geometry and Physics (Florence, 1982)”, p. 71–84, Pitagora, Bologna, 1983.
- [22] D. A. Leites, *Introduction to the theory of supermanifolds*, Russian Math. Surveys **35** (1980), 1–64.
- [23] Yu. I. Manin, *Gauge Field Theory and Complex Geometry*, Springer-Verlag, Berlin, 1988.
- [24] V. Molotkov, *Infinite-dimensional  $\mathbb{Z}_2^k$  supermanifolds*, ICTP preprints IC/84/183 (1984) (scanned version available at <http://ccdb4fs.kek.jp/cgi-bin/allpdf?198506284>)
- [25] J. Monterde and A. Montesinos, *Integral curves of derivations*, Ann. Global Anal. Geom. **6**(2) (1988), 177–189.
- [26] P. van Nieuwenhuizen, *Supergravity*, Phys. Rep. **68**, n.4 (1981), 189–398.
- [27] A. Rogers, *A global theory of supermanifolds*, J. Math. Phys. **21**(6) (1980), 1352–1365.
- [28] A. Rogers, *Graded Manifolds, Supermanifolds and Infinite-Dimensional Grassmann Algebras*, Comm. Math. Phys. **105** (1986), 375–384.
- [29] A. Rogers, *Supermanifolds: Theory and Applications*, World Scientific, New Jersey, 2007.
- [30] C. Sachse, *A Categorical Formulation of Superalgebras and Supergeometry*, preprint posted on ArXiv (2008) (<http://arxiv.org/abs/0802.4067>).

- [31] C. Sachse and C. Wockel, *The diffeomorphism supergroup of a finite-dimensional supermanifold*, preprint posted on arXiv (2009) (<http://arxiv.org/abs/0904.2726>).
- [32] A. Santi and A. Spiro, *Super Poincarè algebras, space-times and supergravities (I)*, preprint posted on ArXiv (2010) (<http://arxiv.org/abs/1011.2722>).
- [33] A. S. Schwarz, *Supergravity, Complex Geometry and G-structures*, Comm. Math. Phys. **87** (1982), 37–63.
- [34] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton University Press, New Jersey, 1992.
- [35] J. Wess and B. Zumino, *Superspace formulation of supergravity*, Phys. Lett. **B 66** (1977), 361–364.

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